Global well-posedness for free boundary problem of the Oldroyd-B Model fluid flow (Mathematical Analysis of Viscous Incompressible Fluid)

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1 Introduction

Let Ω be a bounded domain in N-dimensional Euclidean space $\mathbb{R}^{N}(N\geq 2)$ occupied by a compressible viscous barotropic non-Newtonian fluid of Oldroyd-B model. We assume that the boundary of Ω consists of two parts Γ and S, where $\Gamma\cap S=\emptyset$. Let Ω_{t} and Γ_{t} be time evolutions of Ω and Γ, while S be fixed. We assume that the boundary of Ω_{t} consists of Γ_{t} and S with $\Gamma_{t}\cap S=\emptyset$. Let $\rho:\Omega\times[0,T)\rightarrow \mathbb{R}$, $\mathbf{v}:\Omega\times[0,T)\rightarrow \mathbb{R}^{N}$ and $\tau:[0,\infty)\times\Omega\rightarrow \mathbb{R}^{N\times N}$ be the density field, the velocity field, and the elastic part of the stress tensor, respectively. Then the problem is described by the following system:

$$\begin{cases} \partial_{t}\rho+\mathrm{div}\,(\rho\mathbf{v})=0 & \text{in } \Omega_{t},\\ \rho(\partial_{t}\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v})-\mathrm{Div}\,\mathbf{T}(\mathbf{v},\rho)=\beta\mathrm{Div}\,\tau & \text{in } \Omega_{t},\\ \partial_{t}\tau+\mathbf{v}\cdot\nabla\tau+\gamma\tau=\delta\mathbf{D}(\mathbf{v})+g_{\alpha}(\nabla\mathbf{v},\tau) & \text{in } \Omega_{t},\\ (\mathbf{T}(\mathbf{v},\rho)+\beta\tau)\mathbf{n}_{t}=-P(\rho_{*})\mathbf{n}_{t} & \text{on } \Gamma_{t},\\ \mathbf{v}=0 & \text{on } S,\\ (\rho,\mathbf{v},\tau)|_{t=0}=(\rho_{*}+\theta_{0},\mathbf{v}_{0},\tau_{0}) & \text{in } \Omega_{0},\\ \Omega_{t}|_{t=0}=\Omega_{0},\quad \Gamma_{t}|_{t=0}=\Gamma, \end{cases} \tag{1.1}$$

for $0<t<T$. The mass density of the reference domain Ω that is ρ_{*} is a positive constant, $\mathbf{T}(\mathbf{v},\rho)$ the stress tensor of the form

$$\mathbf{T}(\mathbf{v},\rho)=\mathbf{S}(\mathbf{v})-P(\rho)\mathbf{I}\ \text{ with }\ \mathbf{S}(\mathbf{v})=\mu\mathbf{D}(\mathbf{v})+(\nu-\mu)\mathrm{div}\,\mathbf{v}\mathbf{I}, \tag{1.2}$$

$\mathbf{D}(\mathbf{v})$, $\mathbf{v}=(v_{1},\ldots,v_{N})$, the doubled deformation tensor whose (i,j) components are $D_{ij}(\mathbf{v})=\partial_{i}v_{j}+\partial_{j}v_{i}$ $(\partial_{i}=\partial/\partial x_{j})$, $\mathbf{I}$ the $N\times N$ identity matrix, μ, ν, β, γ and δ are positive constants (μ and ν are the first and second viscosity coefficients, respectively), $\mathbf{n}_{t}$ is the unit outer normal to Γ_{t}, $P(\rho)$ a C^{∞} function defined for $\rho>0$ which satisfies that $P'(\rho)>0$ for $\rho>0$. Moreover, the function $g_{\alpha}(\nabla\mathbf{u},\tau)$ has a form

$$g_{\alpha}(\nabla\mathbf{v},\tau)=\mathbf{W}(\mathbf{v})\tau-\tau\mathbf{W}(\mathbf{v})+\alpha(\tau\mathbf{D}(\mathbf{v})+\mathbf{D}(\mathbf{v})\tau), \tag{1.3}$$

where α is a constant with $-1\leq\alpha\leq 1$ and $\mathbf{W}(\mathbf{v})$ the doubled antisymmetric part of the gradient $\nabla\mathbf{v}$ whose (i,j) components are $W_{ij}(\mathbf{v})=\partial_{i}v_{j}-\partial_{j}v_{i}$. Finally, for any matrix field $\mathbf{K}$ whose components are K_{ij}, the quantity $\mathrm{Div}\,\mathbf{K}$ is an N vector whose i-th component is $\sum_{j=1}^{N}\partial_{j}K_{ij}$, and also $\mathrm{div}\,\mathbf{v}=\sum_{j=1}^{N}\partial_{j}v_{j}$, and $\mathbf{v}\cdot\nabla\mathbf{v}$ is an N vector whose i-th component is $\sum_{j=1}^{N}v_{j}\partial_{j}v_{i}$.

Aside from the dynamical system (1.1), a further kinematic condition for Γ_{t} is satisfied, which gives

$$\Gamma_{t}=\{x\in\mathbb{R}^{N}\mid x=\mathbf{x}(\xi,t)\ \ (\xi\in\Gamma)\}, \tag{1.4}$$

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where $\mathbf{x} = \mathbf{x}(\xi, t)$ is the solution to the Cauchy problem:

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, t) \quad (t > 0), \quad \mathbf{x}|_{t=0} = \xi \in \overline{\Omega}. \tag{1.5}$$

This fact means that the free surface Γ_t consists of the same fluid particles, which do not leave it and are not incident of it from inside Ω_t for $t > 0$. It is clear that $\Omega_t = \{x \in \mathbb{R}^N \mid x = \mathbf{x}(\xi, t) \ \ (\xi \in \Omega)\}$.

Several recent studies investigating the Oldroyd-B model have been carried out by researchers. Preliminary work on incompressible case was undertaken by Oldroyd [5]. He introduced the set of equations in (1.1) in the incompressible viscous fluid case, that is ρ is a positive constant in (1.1). This equation system describe the flow of viscoelastic fluids, which provides a simple linear viscoelastic model for dilute polymer solutions, based on the dumbbell model. After worth, the set of equations in (1.1) is called the Oldroyd-B type fluid.

On the other hand, concerning the study for the compressible case we know only the result about the local wellposedness of non-Newtonian compressible viscous barotropic fluid flow of Oldroyd-B type with free surface due to Maryani [3] in the maximal L_p-L_q regularity class in a bounded domain and some unbounded domains which satisfy some uniformity. This paper is the continuation of Maryani [3] and the global wellposedness of problem (1.1) is proved in the bounded domain case.

Morever, Shibata [8] proved the global well-posedness in a bounded domain also in the maximal L_p-L_q regularity class, assuming that the initial data are small enough and orthogonal to the rigid space. Our idea of proof follows Shibata [8].

The purpose of this paper is to prove the global well-posedness for problem (1.1) in the maximal L_p-L_q regularity class in a bounded domain Ω with $2 < p < \infty$ and $N < q < \infty$, assuming that initial data are small enough and orthogonal to the rigid motion when $S = \emptyset$. To prove it, we use the Lagrange coordinate instead of the Euler coordinate and prolong the local in time solutions in the Lagrange coordinate to any time interval. To do this, the decay properties of solutions play an essential role, which is proved in the case where the velocity field is orthogonal to the rigid motion in the Euler coordinate when $S = \emptyset$. And we formulate this fact in the estimates of solutions to the linearized equations.

Since Ω_t should be decided, we formulate problem (1.1) in the Lagrange coordinates. In fact, if the velocity field $\mathbf{u}(\xi, t)$ is known as a function of the Lagrange coordinates $\xi \in \Omega$, then in view of (1.5) the connection between the Euler coordinates $x \in \Omega_t$ and the Lagrange coordinates $\xi \in \Omega$ is written in the form:

$$x = \xi + \int_0^t \mathbf{u}(\xi, s)ds \equiv \mathbf{X}_{\mathbf{u}}(\xi, t) \tag{1.6}$$

where $\mathbf{u}(\xi, t) = (u_1(\xi, t), \ldots, u_N(\xi, t)) = \mathbf{v}(\mathbf{X}_{\mathbf{u}}(\xi, t), t)$. Let $\mathbf{A}$ be the Jacobi matrix of the transformation $x = \mathbf{X}_{\mathbf{u}}(\xi, t)$ whose (i, j) element is $a_{ij} = \delta_{ij} + \int_0^t (\frac{\partial u_i}{\partial \xi_j})(\xi, s)ds$. There exists a small number σ such that $\mathbf{A}$ is invertible, that is $\det \mathbf{A} \neq 0$, provided that

$$\sup_{0<t<T} \| \int_0^t \nabla \mathbf{u}(\cdot, s)\, ds\|_{L_\infty(\Omega)} \leq \sigma. \tag{1.7}$$

In this case, we have $\nabla_x = \mathbf{A}^{-1}\nabla_\xi = (\mathbf{I} + \mathbf{V}_0(\int_0^t \nabla \mathbf{u}(\xi, s)ds))\nabla_\xi$, where $\mathbf{V}(\mathbf{K})$ is an $N \times N$ matrix of C^∞ functions with respect to $\mathbf{K} = (K_{ij})$ which defined on $|\mathbf{K}| < 2\sigma$. Here, K_{ij} is the corresponding variable to $\int_0^t (\frac{\partial u_i}{\partial \xi_j})(\cdot, s)ds$. We have $\mathbf{V}_0(0) = 0$. Let $\mathbf{n}$ be the unit outward normal to S, and then we have

$$\mathbf{n}_t = \frac{\mathbf{A}^{-1}\mathbf{n}}{|\mathbf{A}^{-1}\mathbf{n}|} \tag{1.8}$$

Let $\rho(x, t)$, $\mathbf{v}(x, t)$ and $\tau(x, t)$ be solutions of (1.1) and let

$$\rho_* + \theta(\xi, t) = \rho(\mathbf{X}_{\mathbf{u}}(\xi, t), t), \ \ \mathbf{u}(\xi, t) = \mathbf{v}(\mathbf{X}_{\mathbf{u}}(\xi, t), t), \ \ \omega(\xi, t) = \tau(\mathbf{X}_{\mathbf{u}}(\xi, t), t). \tag{1.9}$$

And then, problem (1.1) is written in the form:

$$
\begin{cases}
\theta_t + \rho_* \mathrm{div}\,\mathbf{u} = f(\theta, \mathbf{u}, \omega) & \text{in } \Omega \times (0,T), \\
\rho_* \mathbf{u}_t - \mathrm{Div}\,\mathbf{S}(\mathbf{u}) + P'(\rho_*)\nabla\theta - \beta \mathrm{Div}\,\omega = \mathbf{g}(\theta, \mathbf{u}, \omega) & \text{in } \Omega \times (0,T), \\
\omega_t + \gamma\omega - \delta \mathbf{D}(\mathbf{u}) = \mathbf{L}(\theta, \mathbf{u}, \omega) & \text{in } \Omega \times (0,T), \\
(\mathbf{S}(\mathbf{u}) - P'(\rho_*)\theta\mathbf{I} + \beta\omega)\mathbf{n} = \mathbf{h}(\theta, \mathbf{u}, \omega) & \text{on } \Gamma \times (0,T), \\
\mathbf{u} = 0 & \text{on } S \times (0,T), \\
(\theta, \mathbf{u}, \omega)|_{t=0} = (\theta_0, \mathbf{v}_0, \tau_0) & \text{in } \Omega.
\end{cases}
\tag{1.10}
$$

Here, f, $\mathbf{g}$, $\mathbf{L}$ and $\mathbf{h}$ are nonlinear functions define by

$$
f(\theta, \mathbf{u}, \omega) = -\theta \mathrm{div}\,\mathbf{u} - (\rho_* + \theta)\mathbf{V}_{\mathrm{div}}\,(\int_0^t \nabla \mathbf{u} ds)\nabla\mathbf{u} \tag{1.11}
$$

$$
\begin{aligned}
\mathbf{g}(\theta, \mathbf{u}, \omega) = & -\theta\mathbf{u}_t + \mathrm{Div}\left(\mu\mathbf{V}_D(\int_0^t \nabla\mathbf{u} ds)\nabla\mathbf{u} + (\nu - \mu)\mathbf{V}_{\mathrm{div}}\,(\int_0^t \nabla\mathbf{u} ds)\nabla\mathbf{u}\mathbf{I}\right) \\
& + \mathbf{V}_{\mathrm{div}}\,(\int_0^t \nabla\mathbf{u} ds)\nabla\left(\mu(\mathbf{D}(\mathbf{u}) + \mathbf{V}_D(\int_0^t \nabla\mathbf{u} ds)\nabla\mathbf{u}) + (\nu-\mu)(\mathrm{div}\,\mathbf{u} + \mathbf{V}_{\mathrm{div}}\,(\int_0^t \nabla\mathbf{u} ds)\nabla\mathbf{u})\mathbf{I}\right) \\
& - P'(\rho_* + \theta)\mathbf{V}_D(\int_0^t \nabla\mathbf{u} ds)\nabla\theta + \beta\mathbf{V}_{\mathrm{div}}\,(\int_0^t \nabla\mathbf{u} ds)\omega - \nabla\left(\int_0^1 P''(\rho_* + \ell\theta)(1-\ell)d\ell\theta^2\right) \\
\mathbf{L}(\theta, \mathbf{u}, \omega) = & \delta\mathbf{V}_D(\int_0^t \nabla\mathbf{u} ds)\nabla\mathbf{u} + g_\alpha(\nabla\mathbf{u}, \omega) + g_\alpha(\mathbf{V}_w(\int_0^t \nabla\mathbf{u} ds)\nabla\mathbf{u}, \omega) \\
\mathbf{h}(\theta, \mathbf{u}, \omega) = & -\{\mu\mathbf{V}_D(\int_0^t \nabla\mathbf{u} ds)\nabla\mathbf{u} + (\nu-\mu)(\mathbf{V}_{\mathrm{div}}\,(\int_0^t \nabla\mathbf{u} ds)\nabla\mathbf{u})\mathbf{I}\}\mathbf{n} - \beta\omega\mathbf{V}_D(\int_0^t \nabla\mathbf{u} ds)\mathbf{n} \\
& - \{\mu(\mathbf{D}(\mathbf{u}) + \mathbf{V}_D(\int_0^t \nabla\mathbf{u} ds)\nabla\mathbf{u}) + (\nu-\mu)(\mathrm{div}\,\mathbf{u} + \mathbf{V}_{\mathrm{div}}\,(\int_0^t \nabla\mathbf{u} ds)\nabla\mathbf{u})\mathbf{I}\}\mathbf{V}_D(\int_0^t \nabla\mathbf{u} ds)\mathbf{n} \\
& + (\int_0^1 P''(\rho_* + \ell\theta)(1-\ell)d\ell\theta^2)\mathbf{n} + (P(\rho_* + \theta) - P(\rho_*))\mathbf{V}_D(\int_0^t \nabla\mathbf{u} ds)\mathbf{n}.
\end{aligned}
\tag{1.12}
$$

Here $\mathbf{V}_D(\mathbf{K})$, $\mathbf{V}_w(\mathbf{K})$, and $\mathbf{V}_{\mathrm{div}}\,(\mathbf{K})$ are some matrices of C^∞ functions with respect to $\mathbf{K}$ defined on $|\mathbf{K}| \le \sigma$, which satisfy the null condition:

$$
\mathbf{V}_D(0) = 0, \quad \mathbf{V}_w(0) = 0, \quad \mathbf{V}_{\mathrm{div}}\,(0) = 0.
$$

To state our main results, at this stage we introduce our notation used throughout the paper.

Notation $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of all natural numbers, real numbers and complex numbers, respectively. We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $Sym(\mathbb{R}^N)$ and $ASym(\mathbb{R}^N)$ be the set of all $N \times N$ symmetric and anti-symmetric matrices, respectively. For $1 < q < \infty$, let $q' = q/(q-1)$, which is the dual exponent of q and satisfies $1/q + 1/q' = 1$. For any multi-index $\kappa = (\kappa_1, \ldots, \kappa_N) \in \mathbb{N}_0^N$, we write $|\kappa| = \kappa_1 + \cdots + \kappa_N$ and $\partial_x^k = \partial_1^{\kappa_1} \cdots \partial_N^{\kappa_N}$ with $x = (x_1, \ldots, x_N)$. For scalar function f and N-vector of functions $\mathbf{g}$, we set

$$
\nabla f = (\partial_1 f, \ldots, \partial_N f), \ \ \nabla\mathbf{g} = (\partial_i g_j \mid i,j = 1, \ldots, N), \\
\nabla^2 f = \{\partial^\alpha f \mid |\alpha| = 2\}, \ \ \nabla^2\mathbf{g} = \{\partial^\alpha g_i \mid |\alpha| = 2,\, i = 1, \ldots, N\}
$$

For Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from X into Y, and $\mathrm{Hol}\,(\mathrm{U}, \mathcal{L}(\mathrm{X},\mathrm{Y}))$ the set of all $\mathcal{L}(X,Y)$ valued holomorphic functions defined on a domain U in $\mathbb{C}$. For any domain D in $\mathbb{R}^N$ and $1 \le p, q \le \infty$ $L_q(D)$, $W_q^m(D)$, $B_{p,q}^s(D)$ and $H_q^s(D)$ denote the usual Lebesgue space, Sobolev space, Besov space and Bessel potential space, while $\|\cdot\|_{L_q(D)}$, $\|\cdot\|_{W_q^m(D)}$, $\|\cdot\|_{B_{q,p}^s(D)}$ and $\|\cdot\|_{H_q^s(D)}$ denote their norms, respectively. We set $W_q^0(D) = L_q(D)$ and $W_q^s(D) = B_{q,q}^s(D)$. $C^\infty(D)$ denotes the set all C^∞ functions defined on D. $L_p((a,b),X)$ and $W_p^m((a,b),X)$ denote the usual Lebesgue space and Sobolev space of X-valued function defined on an interval (a,b), while $\|\cdot\|_{L_p((a,b),X)}$ and $\|\cdot\|_{W_p^m((a,b),X)}$ denote their norms, respectively. Moreover, we set

$$
\|e^{\eta t} f\|_{L_p((a,b),X)} = \left(\int_a^b (e^{\eta t}\|f(t)\|_X)^p dt\right)^{1/p} \quad \text{for } 1 \le p < \infty.
$$

The d-product space of X is defined by $X^d = \{f = (f, \dots, f_d) \mid f_i \in X\ (i = 1, \dots, d)\}$, while its norm is denoted by $\|\cdot\|_X$ instead of $\|\cdot\|_{X^d}$ for the sake of simplicity. We set

$$W_q^{m,\ell}(D) = \{(f, \mathbf{g}, \mathbf{H}) \mid f \in W_q^m(D),\ \ \mathbf{g} \in W_q^\ell(D)^N,\ \ \mathbf{H} \in W_q^m(D)^{N\times N}\},$$
$$\|(f, \mathbf{g}, \mathbf{H})\|_{W_q^{m,\ell}(\Omega)} = \|(f, \mathbf{H})\|_{W_q^m(\Omega)} + \|\mathbf{g}\|_{W_q^\ell(\Omega)}.$$

For $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$, we set $\mathbf{a}\cdot\mathbf{b} =< \mathbf{a}, \mathbf{b} >= \sum_{j=1}^n a_j b_j$. For scalar functions f, g and N-vectors of functions $\mathbf{f}$, $\mathbf{g}$ we set $(f, g)_D = \int_D fg\,dx$, $(\mathbf{f}, \mathbf{g})_D = \int_D \mathbf{f}\cdot\mathbf{g}\,dx$, $(f, g)_\Gamma = \int_\Gamma fg\,d\sigma$, $(\mathbf{f}, \mathbf{g})_\Gamma = \int_\Gamma \mathbf{f}\cdot\mathbf{g}d\sigma$, where σ is the surface element of Γ. For $N\times N$ matrices of functions $\mathbf{A} = (A_{ij})$ and $\mathbf{B} = (\mathbf{B}_{ij})$, we set $(\mathbf{A}, \mathbf{B})_D = \int_D \mathbf{A} : \mathbf{B}\,dx$ and $(\mathbf{A}, \mathbf{B})_\Gamma = \int_\Gamma \mathbf{A} : \mathbf{B}\,d\sigma$, where $\mathbf{A} : \mathbf{B} \equiv \sum_{i,j=1}^N A_{ij}B_{ij}$. The letter C denotes generic constants and the constant $C_{a,b,\dots}$ depends on $a, b, \dots$. The values of constants C and $C_{a,b,\dots}$ may change from line to line. We use small boldface letters, e.g. $\mathbf{u}$ to denote vector-valued functions and capital boldface letters, e.g. $\mathbf{H}$ to denote matrix-valued functions, respectively. But, we also use the Greek letters, e.g. ρ, θ, τ, ω, to denote mass densities, and elastic tensors unless the confusion may occur, although they are $N\times N$ matrices.

To state the compatibility condition for initial data θ_0, $\mathbf{v}_0$, and τ_0, we introduce the space $\mathcal{D}_{q,p}(\Omega)$ defined by

$$\begin{aligned}\mathcal{D}_{q,p}(\Omega) = \Big\{&(\theta_0, \mathbf{v}_0, \tau_0) \in W_q^1(\Omega) \times B_{q,p}^{2(1-1/p)}(\Omega)^N \times W_q^1(\Omega)^{N\times N} \mid \\ &(\mathbf{S}(\mathbf{v}_0) - (P(\rho_* + \theta_0) - P(\rho_*))\mathbf{I} + \beta\tau_0)\mathbf{n} = 0 \text{ on } \Gamma, \quad \mathbf{v}_0|_S = 0\Big\}.\end{aligned} \tag{1.13}$$

For the notational simplicity, we set

$$\|(\theta_0, \mathbf{v}_0, \tau_0)\|_{\mathcal{D}_{q,p}(\Omega)} = \|\theta_0\|_{W_q^1(\Omega)} + \|\mathbf{v}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|\tau_0\|_{W_q^1(\Omega)}.$$

The following theorem about the local well-posedness of problem (1.10) was proved by Maryani [3].

Theorem 1.1. *Let $N < q < \infty$, $2 < p < \infty$ and $R > 0$. Assume that Γ and S are $W_q^{2-1/q}$ compact hyper-surfaces. Then, there exists a time $T = T(R) > 0$ such that for any initial data $(\theta_0, \mathbf{v}_0, \tau_0) \in \mathcal{D}_{q,p}(\Omega)$ satisfying the conditions:*

$$\frac{2}{3}\rho_* < \rho_* + \theta_0(x) < \frac{4}{3}\rho_* \quad (x \in \Omega) \tag{1.14}$$

and

$$\|(\theta_0, \mathbf{v}_0, \tau_0)\|_{\mathcal{D}_{q,p}(\Omega)} \leq R \tag{1.15}$$

problem of (1.10) *admits a unique solution* $(\theta, \mathbf{u}, \omega)$ *with*

$$\theta \in W_p^1((0,T), W_q^1(\Omega)), \quad \mathbf{u} \in W_p^1((0,T), L_q(\Omega)) \cap L_p((0,T), W_q^2(\Omega)), \quad \omega \in W_p^1((0,T), W_q^1(\Omega))$$

satisfying (1.7), *the range condition:* $\frac{1}{3}\rho_* < \rho_* + \theta(x,t) < \frac{5}{3}\rho_*$ *for any* $(x,t) \in \Omega \times (0,T)$ *and possessing the estimate:*

$$\|\theta\|_{W_p^1((0,t),W_q^1(\Omega))} + \|\mathbf{u}\|_{W_p^1((0,t),L_q(\Omega))} + \|\mathbf{u}\|_{L_p((0,t),W_q^2(\Omega))} + \|\omega\|_{W_p^1((0,t),W_q^1(\Omega))} \leq C(R)$$

Remark 1.2. (1) The range condition (1.14) follows from $\|\theta_0\|_{L_\infty(\Omega)} \leq \frac{\rho_*}{3}$.

(2) The local well-posedness was proved under the assumption that Ω is a uniform $W_q^{2-1/q}$ domain in [3]. And, if Γ and S are compact $W_q^{2-1/q}$ hyper-surfaces, then Ω is a uniform $W_q^{2-1/q}$ domain.

(3) By using the uniqueness of solutions, we see that if $\tau_0(x) \in Sym(\mathbb{R}^N)$ for almost all $x \in \Omega$ then $\omega(x,t) \in Sym(\mathbb{R}^N)$ for almost all $(x,t) \in \Omega \times (0,\infty)$, too.

In order to state the global well-posedness of problem (1.10), we introduce the rigid space $\mathcal{R}_d$ which is defined by

$$\mathcal{R}_d = \{\mathbf{A}x + \mathbf{b} \mid \mathbf{A} \in ASym(\mathbb{R}^N) \text{ and } \mathbf{b} \in \mathbb{R}^N\}. \tag{1.16}$$

Let $\{\mathbf{p}_\ell\}_{\ell=1}^M$ be the system of orthonormal basis of $\mathcal{R}_d$.

The following theorem is our main result concerning the global well-posedness of problem (1.10).

Theorem 1.3. *Let $N < q < \infty$ and $2 < p < \infty$. Let ℓ_b be a number such that $\ell_b = 3$ when $S \neq \emptyset$ and $\ell_b = 2$ when $S = \emptyset$. Assume that S and Γ are $W_q^{\ell_b - 1/q}$ compact hyper-surfaces and that $\Gamma \neq \emptyset$. Assume that the viscosity coefficients μ and ν satisfy the stability condition:*

$$\mu > 0, \quad \nu > \frac{N-2}{N}\mu \tag{1.17}$$

Then, there exist positive numbers ϵ and η such that for any initial data $(\theta_0, \mathbf{v}_0, \tau_0) \in \mathcal{D}_{q,p}(\Omega)$ satisfying the condition that $\tau_0(x) \in Sym(\mathbb{R}^N)$ for any $x \in \Omega$, the smallness condition: $\|(\theta_0, \mathbf{v}_0, \tau_0)\|_{\mathcal{D}_{q,p}(\Omega)} \leq \epsilon$ and the orthogonal condition:

$$((\rho_* + \theta_0)\mathbf{v}_0, \mathbf{p}_\ell)_\Omega = 0 \ \text{ for } \ell = 1, \ldots, M \text{ when } S = \emptyset, \tag{1.18}$$

problem (1.10) with $T = \infty$ admits unique solutions θ, $\mathbf{u}$ and ω with

$$\theta \in W_p^1((0,\infty), W_q^1(\Omega)), \ \ \mathbf{u} \in L_p((0,\infty), W_q^2(\Omega)^N) \cap W_p((0,\infty), L_q(\Omega)^N), \ \ \omega \in W_p^1((0,\infty), W_q^1(\Omega)).$$

Moreover, there exists a positive constant γ_0 such that $(\theta, \mathbf{u}, \omega)$ satisfies the estimate:

$$\begin{aligned}&\|e^{\gamma s}(\partial_s\theta, \theta)\|_{L_p((0,t),W_q^1(\Omega))} + \|e^{\gamma s}\partial_s\mathbf{u}\|_{L_p((0,t),L_q(\Omega))} + \|e^{\gamma s}\mathbf{u}\|_{L_p((0,t),W_q^2(\Omega))}\\ &\quad + \|e^{\gamma s}(\partial_s\omega, \omega)\|_{L_p((0,t),W_q^1(\Omega))} \leq C_\gamma \epsilon\end{aligned}$$

for any $t > 0$ and $\gamma \in (0, \gamma_0)$ with some positive number C_γ independent of ϵ and t.

Remark 1.4. Using the argumentation due to Ströhmer [10], we see that the map $x = \mathbf{X}_\mathbf{u}(\xi, t)$ is bijective from Ω onto $\Omega_t = \{x = \mathbf{X}_\mathbf{u}(\xi, t) \mid \xi \in \Omega\}$ with suitable regularity. Therefore, from Theorem 1.3 we have the global well-posedness for problem (1.1).

2 Some decay properties of solutions to the linearized problem

Let Ω be a bounded domain and let both of its boundaries S and Γ be $W_r^{2-1/r}$ hyper-surfaces with $N < r < \infty$, and let q be an exponent such that $1 < q < \infty$ and $\max(q, q') \leq r$. In this section, we show some exponential stability of solutions to the following problem :

$$\begin{cases} \partial_t\theta + \rho_*\mathrm{div}\,\mathbf{u} = f & \text{in } \Omega \times (0,T),\\ \rho_*\partial_t\mathbf{u} - \mathrm{Div}\,\mathbf{S}(\mathbf{u}) + P'(\rho_*)\nabla\theta - \beta\mathrm{Div}\,\tau = \mathbf{g} & \text{in } \Omega \times (0,T),\\ \partial_t\tau + \gamma\tau - \delta\mathbf{D}(\mathbf{u}) = \mathbf{H} & \text{in } \Omega \times (0,T),\\ \mathbf{u} = 0 & \text{on } S \times (0,T),\\ (\mathbf{S}(\mathbf{u}) - P'(\rho_*)\theta\mathbf{I} + \beta\tau)\mathbf{n} = \mathbf{k} & \text{on } \Gamma \times (0,T),\\ (\theta, \mathbf{u}, \tau)|_{t=0} = (\theta_0, \mathbf{u}_0, \tau_0) & \text{in } \Omega.\end{cases} \tag{2.1}$$

For this purpose, first we analyze the coresponding generalized resolvent problem:

$$\begin{cases} \lambda\theta + \rho_*\mathrm{div}\,\mathbf{u} = f & \text{in } \Omega,\\ \rho_*\lambda\mathbf{u} - \mathrm{Div}\,\mathbf{S}(\mathbf{u}) + P'(\rho_*)\nabla\theta - \beta\mathrm{Div}\,\tau = \mathbf{g} & \text{in } \Omega,\\ \lambda\tau + \gamma\tau - \delta\mathbf{D}(\mathbf{u}) = \mathbf{H} & \text{in } \Omega,\\ \mathbf{u} = 0 & \text{on } S,\\ (\mathbf{S}(\mathbf{u}) - P'(\rho_*)\theta\mathbf{I} + \beta\tau)\mathbf{n} = \mathbf{k} & \text{on } \Gamma.\end{cases} \tag{2.2}$$

To quote some results due to Maryani [3], we introduce the $\mathcal{R}$-boundedness of operator families and the Weis operator valued Fourier multiplier theorem.

Definition 2.1. A family of operators $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X,Y)$, if there exist constants $C > 0$ and $p \in [1,\infty)$ such that for any $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{f_j\}_{j=1}^n \subset X$ and sequences $\{r_j\}_{j=1}^n$ of independent, symmetric, $\{-1,1\}$-valued random variables on $[0,1]$, we have the inequality:

$$\left\{\int_0^1 \|\sum_{j=1}^n r_j(u)T_jx_j\|_Y^p\,du\right\}^{1/p} \le C\left\{\int_0^1 \|\sum_{j=1}^n r_j(u)x_j\|_X^p\,du\right\}^{1/p}.$$

The smallest such C is called $\mathcal{R}$-bounded of $\mathcal{T}$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

Let $\mathcal{D}(\mathbb{R},X)$ and $\mathcal{S}(\mathbb{R},X)$ be the set of all X valued C^∞ functions having compact support and the Schwartz space of rapidly decreasing X valued functions, respectively, while $\mathcal{S}'(\mathbb{R},X) = \mathcal{L}(\mathcal{S}(\mathbb{R},\mathbb{C}),X)$. Given $M \in L_{1,\mathrm{loc}}(\mathbb{R}\backslash\{0\},X)$, we define the operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R},X) \to \mathcal{S}'(\mathbb{R},Y)$ by

$$T_M\phi = \mathcal{F}^{-1}[M\mathcal{F}[\phi]], \quad (\mathcal{F}[\phi] \in \mathcal{D}(\mathbb{R},X)). \tag{2.3}$$

The following theorem is obtained by Weis [11].

Theorem 2.2. *Let X and Y be two UMD Banach spaces and $1 < p < \infty$. Let M be a function in $C^1(\mathbb{R}\backslash\{0\},\mathcal{L}(X,Y))$ such that*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\{(\tau\frac{d}{d\tau})^\ell M(\tau) \mid \tau \in \mathbb{R}\backslash\{0\}\}) \le \kappa < \infty \quad (\ell = 0,1)$$

with some constant κ. Then, the operator T_M defined in (2.3) is extended to a bounded linear operator from $L_p(\mathbb{R},X)$ into $L_p(\mathbb{R},Y)$. Moreover, denoting this extension by T_M, we have

$$\|T_M\|_{\mathcal{L}(L_p(\mathbb{R},X),L_p(\mathbb{R},Y))} \le C\kappa$$

for some positive constant C depending on p, X and Y.

Remark 2.3. For the definition of UMD space, we refer to a book due to Amann [1]. For $1 < q < \infty$, Lebesgue space $L_q(\Omega)$ and Sobolev space $W_q^m(\Omega)$ are both UMD spaces.

The resolvent parameter λ in problem (2.2) varies in $\Sigma_{\epsilon,\lambda_0}$ with $\Sigma_{\epsilon,\lambda_0} = \{\lambda \in \mathbb{C} \mid |\arg\lambda| \le \pi-\epsilon,\ |\lambda| \ge \lambda_0\}$ $(\epsilon \in (0,\pi/2), \lambda_0 > 0)$. To quote some unique existence theorem for problem (2.1), we introduce the space $\mathbf{W}_q^{-1}(\Omega)$. Let ι be the extension map $\iota : L_{1,\mathrm{loc}}(\Omega) \to L_{1,\mathrm{loc}}(\mathbb{R}^N)$ having the following properties :

1. For any $1 < q < \infty$ and $f \in W_q^1(\Omega)$, $\iota f \in W_q^1(\mathbb{R}^n)$, $\iota f = f$ in Ω and $\|\iota f\|_{W_q^i(\mathbb{R}^n)} \le C\|f\|_{W_q^i(\Omega)}$ for $i = 0,1$ with some constant C depending on q, r and Ω.
2. For any $1 < q < \infty$ and $f \in W_q^1(\Omega)$, $\|\iota(\nabla f)\|_{H_q^{-1}(\mathbb{R}^N)} \le C\|f\|_{L_q(\Omega)}$ with some constant C depending on q, r and Ω.

Then, $\mathbf{W}_q^{-1}(\Omega)$ is defined by

$$\mathbf{W}_q^{-1}(\Omega) = \{f \in L_{1,\mathrm{loc}}(\Omega) \mid \|f\|_{\mathbf{W}_q^{-1}(\Omega)} = \|\iota f\|_{H_q^{-1}(\mathbb{R}^N)} < \infty\}.$$

According to Maryani [3], we have

Theorem 2.4. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$ and $N < r < \infty$. Assume that $r \ge \max(q,q')$. Let Ω be a bounded domain in $\mathbb{R}^N$, whose boundaries S and Γ are both $W_r^{2-1/r}$ compact hyper-surfaces. Let*

$$\Sigma_{\epsilon,\lambda_0} = \{\lambda \in \mathbb{C}\setminus\{0\} \mid |\arg\lambda| \le \pi-\epsilon,\ |\lambda| \ge \lambda_0\}.$$

Let

$$\begin{aligned} X_q(\Omega) &= \{(f,\mathbf{g},\mathbf{H},\mathbf{k}) \mid (f,\mathbf{g},\mathbf{H}) \in W_q^{1,0}(\Omega), \mathbf{k} \in W_q^1(\Omega)^N\}, \\ \mathcal{X}_q(\Omega) &= \{(F_1,\mathbf{F}_2,\mathbf{F}_3,\mathbf{F}_4,\mathbf{F}_5) \mid \\ &\qquad F_1 \in W_q^1(\Omega), \mathbf{F}_2 \in L_q(\Omega)^N, \mathbf{F}_3 \in L_q(\Omega)^N, \mathbf{F}_4 \in L_q(\Omega)^{N^2}, \mathbf{F}_5 \in W_q^1(\Omega)^{N^2}\}. \end{aligned}$$

Then, there exists a $\lambda_0 \geq 1$ *and an operator family* $R(\lambda) \in \mathrm{Hol}(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega), \mathrm{W}^{1,2}_q(\Omega)))$ *such that for any* $(f, \mathbf{g}, \mathbf{H}, \mathbf{k}) \in X_q(\Omega)$ *and* $\lambda \in \Sigma_{\epsilon,\lambda_0}$, $(\rho, \mathbf{u}, \tau) = R(\lambda)(f, \mathbf{g}, \lambda^{1/2}\mathbf{k}, \nabla\mathbf{k}, \mathbf{H})$ *is a unique solution to problem* (2.2).

Moreover, there exists a constant C *such that*

$$\begin{aligned}
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), W^{1,0}_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\lambda R(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) &\leq C \quad (\ell = 0, 1),\\
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), W^{1,0}_q(\Omega))}(\{(\tau\partial_\tau)^\ell(\gamma R(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) &\leq C \quad (\ell = 0, 1),\\
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), L_q(\Omega)^{N^2})}(\{(\tau\partial_\tau)^\ell(\lambda^{1/2}\nabla P_v R(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) &\leq C \quad (\ell = 0, 1),\\
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), L_q(\Omega)^{N^3})}(\{(\tau\partial_\tau)^\ell(\nabla^2 P_v R(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) &\leq C \quad (\ell = 0, 1),
\end{aligned} \tag{2.4}$$

with $\lambda = \gamma + i\tau$. *Here,* P_v *is the projection operator defined by* $P_v(\rho, \mathbf{u}, \tau) = \mathbf{u}$.

Remark 2.5. (1) The F_1, $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_4$ and $\mathbf{F}_5$ are variables corresponding to f, $\mathbf{g}$, $\lambda^{1/2}\mathbf{k}$, $\nabla\mathbf{k}$, and $\mathbf{H}$, respectively.
(2) Theorem 2.4 was proved in [3], where the same problem was treated even in the unbounded domain case.

As was shown in [3], applying Theorem 2.4 with the help of Theorem 2.2, we have

Theorem 2.6. *Let* $1 < p, q < \infty$, $N < r < \infty$ *and* $T > 0$. *Assume that* $\max(q, q') \leq r$. *Let* Ω *be a bounded domain in* $\mathbb{R}^N$, *whose boundaries* S *and* Γ *are both* $W^{2-1/r}_r$ *compact hyper-surfaces. Then, there exists a positive number* η_0 *such that for any initial data* $(\theta_0, \mathbf{u}_0, \tau_0) \in W^1_q(\Omega) \times B^{2(1-1/p)}_{q,p}(\Omega)^N \times W^1_q(\Omega)^{N\times N}$ *and right-hand sides* f, $\mathbf{g}$, $\mathbf{H}$ *and* $\mathbf{k}$ *with*

$$(f, \mathbf{g}, \mathbf{H}) \in L_p((0,T), W^{1,0}_q(\Omega)), \;\; \mathbf{k} \in L_p((0,T), W^1_q(\Omega)^N) \cap W^1_p((0,T), \mathbf{W}^{-1}_q(\Omega)^N) \tag{2.5}$$

satisfying the compatibility condition:

$$(\mathbf{S}(\mathbf{u}_0) - P'(\rho_*)\theta_0\mathbf{I} + \beta\tau_0)\mathbf{n} = \mathbf{k}|_{t=0} \;\; \text{on } \Gamma, \quad \mathbf{u}_0 = 0 \;\; \text{on } S, \tag{2.6}$$

problem (2.1) *admits unique solutions* θ, $\mathbf{u}$, *and* τ *with*

$$\theta \in W^1_p((0,T), W^1_q(\Omega)), \;\; \mathbf{u} \in L_p((0,T), W^2_q(\Omega)^N) \cap W^1_p((0,T), L_q(\Omega)^N), \;\; \tau \in W^1_p((0,T), W^1_q(\Omega)^{N\times N})$$

possessing the estimate:

$$\begin{aligned}
&\|\theta\|_{W^1_p((0,t),W^1_q(\Omega))} + \|\partial_s\mathbf{u}\|_{L_p((0,t),L_q(\Omega))} + \|\mathbf{u}\|_{L_p((0,t),W^2_q(\Omega))} + \|\tau\|_{W^1_p((0,t),W^1_q(\Omega))}\\
&\quad \leq C_\gamma e^{\eta_0 t}\{\|\theta_0\|_{W^1_q(\Omega)} + \|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|\tau_0\|_{W^1_q(\Omega)}\\
&\qquad + \|(f, \mathbf{H}, \mathbf{k})\|_{L_p((0,t),W^1_q(\Omega))} + \|\mathbf{g}\|_{L_p((0,t),L_q(\Omega))} + \|\partial_s\mathbf{k}\|_{L_p((0,t),\mathbf{W}^{-1}_q(\Omega))}\}
\end{aligned}$$

for any $t \in (0, T)$ *with some constant* C *independent of* t.

To prove the global well-posedness of problem (1.10), we need some decay properties of solutions to (2.1), which is stated as follows:

Theorem 2.7. *Let* $1 < p, q < \infty$, $N < r < \infty$ *and* $T > 0$. *Assume that* $\max(q, q') \leq r$. *Let* ℓ_b *be the number defined in Theorem 1.1. Let* Ω *be a bounded domain in* $\mathbb{R}^N$, *whose boundaries* S *and* Γ *are both* $W^{\ell_b-1/r}_r$ *compact hyper-surfaces. Then, for any initial data* $(\theta_0, \mathbf{u}_0, \tau_0) \in W^1_q(\Omega) \times B^{2(1-1/p)}_{q,p}(\Omega)^N \times W^1_q(\Omega)^{N\times N}$ *and right-hand sides* f, $\mathbf{g}$, $\mathbf{H}$ *and* $\mathbf{k}$ *satisfying* (2.5), *the compatibility condition* (2.6), *and the symmetric condition:* $\tau_0(x) \in Sym(\mathbb{R}^N)$ *for almost all* $x \in \Omega$, *problem* (2.1) *admits unique solutions* θ, $\mathbf{u}$, *and* τ *with*

$$\theta \in W^1_p((0,T), W^1_q(\Omega)), \;\; \mathbf{u} \in L_p((0,T), W^2_q(\Omega)^N) \cap W_p((0,T), L_q(\Omega)^N), \;\; \tau \in W^1_p((0,T), W^1_q(\Omega)^{N\times N})$$

possessing the estimate :

$$
\begin{aligned}
&\|e^{\eta_1 s}\theta\|_{W^1_p((0,t),W^1_q(\Omega))} + \|e^{\eta_1 s}\partial_s \mathbf{u}\|_{L_p((0,t),L_q(\Omega))} + \|e^{\eta_1 s}\mathbf{u}\|_{L_p((0,t),W^2_q(\Omega))} + \|e^{\eta_1 s}\tau\|_{W^1_p((0,t),W^1_q(\Omega))} \\
&\quad \leq C\Big\{\|\theta_0\|_{W^1_q(\Omega)} + \|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|\tau_0\|_{W^1_q(\Omega)} + \|e^{\eta_1 s}(f,\mathbf{g},\mathbf{H})\|_{L_p((0,t),W^{1,0}_q(\Omega))} \qquad (2.7)\\
&\quad + \|e^{\eta_1 s}\mathbf{k}\|_{L_p((0,t),W^1_q(\Omega))} + \|e^{\eta_1 s}\partial_s\mathbf{k}\|_{L_p((0,t),\mathbf{W}^{-1}_q(\Omega))} + d(S)\sum_{\ell=1}^M\Big(\int_0^t (e^{\eta_1 s}|(\mathbf{u}(\cdot,s),\mathbf{p}_\ell)_\Omega|)^p ds\Big)^{\frac1p}\Big\}
\end{aligned}
$$

for any $t \in (0,T)$ with some positive constants C and η_1. Here, $d(S)$ is the number such that $d(S) = 1$ when $S = \emptyset$ and $d(S) = 0$ when $S \neq \emptyset$.

Remark 2.8. The symmetric condition: $\tau_0(x) \in Sym(\mathbb{R}^N)$ for almost all $x \in \Omega$ implies that $\tau(x,t) \in Sym(\mathbb{R}^N)$ for almost all $(x,t) \in \Omega \times (0,T)$.

To prove Theorem 2.7, first we consider problem (2.1) with $f = 0$, $\mathbf{g} = 0$, $\mathbf{H} = 0$ and $\mathbf{k} = 0$. And then, the corresponding resolvent equation is:

$$
\begin{cases}
\lambda\theta + \rho_*\mathrm{div}\,\mathbf{u} = f & \text{in } \Omega,\\
\rho_*\lambda\mathbf{u} - \mathrm{Div}\,\mathbf{S}(\mathbf{u}) + P'(\rho_*)\nabla\theta - \beta\mathrm{Div}\,\tau = \mathbf{g} & \text{in } \Omega,\\
\lambda\tau + \gamma\tau - \delta\mathbf{D}(\mathbf{u}) = \mathbf{H} & \text{in } \Omega,\\
\mathbf{u} = 0 & \text{on } S,\\
(\mathbf{S}(\mathbf{u}) - P'(\rho_*)\theta\mathbf{I} + \beta\tau)\mathbf{n} = 0 & \text{on } \Gamma,
\end{cases} \qquad (2.8)
$$

where θ_0, $\mathbf{u}_0$ and τ_0 have been renamed f, $\mathbf{g}$ and $\mathbf{H}$, respectively. We consider problem (2.8) on the underlying space $\mathcal{H}_q(\Omega)$ which is the set of all $(f,\mathbf{g},\mathbf{H}) \in W^{1,0}_q(\Omega)$ such that $\mathbf{g}$ satisfies the orthogonal condition:

$$(\mathbf{g},\mathbf{p}_\ell)_\Omega = 0 \quad (\ell = 1,\ldots,M) \qquad (2.9)$$

when $S = \emptyset$. Note that any solution $(\theta,\mathbf{u},\tau)$ of problem (2.8) satisfies the orthogonal condition:

$$(\mathbf{u},\mathbf{p}_\ell)_\Omega = 0 \quad (\ell = 1,\ldots,M) \qquad (2.10)$$

when $S = \emptyset$. In fact, by the divergence theorem of Gauß, we have

$$
\begin{aligned}
\rho_*\lambda(\mathbf{u},\mathbf{p}_\ell)_\Omega &= (\mathbf{g},\mathbf{p}_\ell)_\Omega + (\mathbf{k},\mathbf{p}_\ell)_\Gamma - \frac{\mu}{2}(\mathbf{D}(\mathbf{u}),\mathbf{D}(\mathbf{p}_\ell))_\Omega\\
&\quad - ((\nu-\mu)\mathrm{div}\,\mathbf{u} - P'(\rho_*)\theta, \mathrm{div}\,\mathbf{p}_\ell)_\Omega - \frac{\beta}{2}(\tau,\mathbf{D}(\mathbf{p}_\ell))_\Omega.
\end{aligned}
$$

Since it holds that

$$\mathbf{D}(\mathbf{p}_\ell) = 0, \quad \mathrm{div}\,\mathbf{p}_\ell = 0 \quad (\ell = 1,\ldots,M), \qquad (2.11)$$

(2.9) implies (2.10).

Let $\dot{W}^2_q(\Omega)^N$ be the set of all $\mathbf{u} \in W^2_q(\Omega)^N$ which satisfies (2.10). And also, we introduce an operator $\mathcal{A}$ and a space $\mathcal{D}_q(\mathcal{A})$ by

$$
\begin{aligned}
&\mathcal{A}(\theta,\mathbf{u},\tau) = (-\rho_*\mathrm{div}\,\mathbf{u}, \rho_*^{-1}(\mathrm{Div}\,\mathbf{S}(\mathbf{u}) - P'(\rho_*)\nabla\theta + \beta\mathrm{Div}\,\tau), -\gamma\tau + \delta\mathbf{D}(\mathbf{u})) \text{ for } (\theta,\mathbf{u},\tau) \in \mathcal{D}_q(\mathcal{A}),\\
&\mathcal{D}_q(\mathcal{A}) = \{(\theta,\mathbf{u},\tau) \in \mathcal{H}_q(\Omega) \mid \mathbf{u} \in \dot{W}^2_q(\Omega)^N,\ \mathbf{u}|_S = 0,\ (\mathbf{S}(\mathbf{u}) - P'(\rho_*)\theta\mathbf{I} + \beta\tau)\mathbf{n}|_\Gamma = 0\}.
\end{aligned}
$$

By using $\mathcal{A}$, problem (2.1) with $f = 0$, $\mathbf{g} = 0$, $\mathbf{H} = 0$ and $\mathbf{k} = 0$ is written in the form:

$$\partial_t(\theta,\mathbf{u},\tau) - \mathcal{A}(\theta,\mathbf{u},\tau) = (0,0,0) \text{ for } t > 0, \quad (\theta,\mathbf{u},\tau)|_{t=0} = (\theta_0,\mathbf{u}_0,\tau_0). \qquad (2.12)$$

Since $\mathcal{R}$ boundedness implies the usual boundedness by choosing $n = 1$ in Definition 2.1, for any $\epsilon \in (0,\pi/2)$ there exists a constant $\lambda_1 > 0$ such that for any $\lambda \in \Sigma_{\epsilon,\lambda_1}$ and $(f,\mathbf{g},\mathbf{H}) \in \mathcal{H}_q(\Omega)$ problem (2.8) admits a unique solution $(\theta,\mathbf{u},\tau) \in \mathcal{D}_q(\Omega)$ possessing the estimate:

$$|\lambda|\|(\theta,\mathbf{u},\tau)\|_{W^{1,0}_q(\Omega)} + \|\mathbf{u}\|_{W^2_q(\Omega)} \leq C\|(f,\mathbf{g},\mathbf{H})\|_{W^{1,0}_q(\Omega)} \qquad (2.13)$$

for any $\lambda \in \Sigma_{\epsilon,\lambda_1}$. Here, we used the fact that (2.9) implies (2.10).

By (2.13), we know that there exists a continuous semigroup $\{T(t)\}_{t\geq0}$ on $\mathcal{H}_q(\Omega)$ associated with problem (2.12) which is analytic. To prove the exponential stability of $\{T(t)\}_{t\geq0}$, it is sufficient to prove

Theorem 2.9. *Let $1 < q < \infty$, $N < r < \infty$ and $\lambda_1 > 0$. Assume that $max(q, q') \leq r$. Let ℓ_b be the number given in Theorem 1.1 and let λ_1 be the number given above. Let Ω be a bounded domain in $\mathbb{R}^N$, whose boundaries S and Γ are both $W_r^{\ell_b-1/r}$ compact hyper-surfaces. Assume that*

$$\mu > 0, \quad \nu > \frac{N-2}{N}\mu. \tag{2.14}$$

Then, for any $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq 0$ and $|\lambda| \leq \lambda_1$ and $(f, \mathbf{g}, \mathbf{H}) \in \mathcal{H}_q(\Omega)$, problem (2.2) with $\mathbf{k} = 0$ admits a unique solution $(\theta, \mathbf{u}, \tau) \in \mathcal{D}_q(\mathcal{A})$ possessing the estimate:

$$\|(\theta, \mathbf{u}, \tau)\|_{W_q^{1,2}(\Omega)} \leq C\|(f, \mathbf{g}, \mathbf{H})\|_{W_q^{1,0}(\Omega)}. \tag{2.15}$$

We postpone the proof of Theorem 2.9 to Sect. 3. By Theorem 2.9, we have

Corollary 2.10. *Let $1 < q < \infty$, $N < r < \infty$ and $\lambda_1 > 0$. Assume that $max(q, q') \leq r$. Let ℓ_b be the number given in Theorem 1.1. Let Ω be a bounded domain in $\mathbb{R}^N$, whose boundaries S and Γ are both $W_r^{\ell_b-1/r}$ compact hyper-surfaces. Assume the condition (2.14) holds. Then, the semigroup $\{T(t)\}_{t\geq 0}$ is exponantially stable on $\mathcal{H}_q(\Omega)$, that is,*

$$\|T(t)(f, \mathbf{g}, \mathbf{H})\|_{W_q^{1,0}(\Omega)} \leq Ce^{-\eta_1 t}\|(f, \mathbf{g}, \mathbf{H})\|_{W_q^{1,0}(\Omega)} \tag{2.16}$$

for any $(f, \mathbf{g}, \mathbf{H}) \in \mathcal{H}_q(\Omega)$ and $t > 0$ with some positive constants C and η_1.

Now, we are in position to prove Theorem 2.7. To reduce the problem to the semigroup setting, first we consider the time shifted equations :

$$\begin{cases} \partial_t\theta + \lambda_0\theta + \rho_*\mathrm{div}\,\mathbf{u} = f & \text{in } \Omega \times (0, T), \\ \rho_*(\partial_t\mathbf{u} + \lambda_0\mathbf{u}) - \mathrm{Div}\,\mathbf{S}(\mathbf{u}) + P'(\rho_*)\nabla\theta - \beta\mathrm{Div}\,\tau = \mathbf{g} & \text{in } \Omega \times (0, T), \\ \partial_t\tau + \gamma\tau + \lambda_0\tau - \delta D(\mathbf{u}) = \mathbf{H} & \text{in } \Omega \times (0, T), \\ \mathbf{u} = 0 & \text{on } S \times (0, T), \\ (\mathbf{S}(\mathbf{u} - P'(\rho_*)\theta\mathbf{I} + \beta\tau) \cdot \mathbf{n} = \mathbf{k} & \text{on } \Gamma_1 \times (0, T), \\ (\theta, \mathbf{u}, \tau)|_{t=0} = (\theta_0, \mathbf{u}_0, \tau_0) & \text{in } \Omega, \end{cases} \tag{2.17}$$

with large $\lambda_0 > 0$. For example, in the case $(\theta_0, \mathbf{u}_0, \tau_0) = (0, 0, 0)$, by using the $\mathcal{R}$-bounded solution operators $R(\lambda)$ given in Theorem 2.4), the solutions of (2.17) is written by the Laplace inverse transform of $R(\lambda + \lambda_0)(\hat{f}(\lambda), \hat{\mathbf{g}}(\lambda), \hat{\mathbf{H}}(\lambda))$, where $\hat{f}(\lambda)$, $\hat{\mathbf{g}}(\lambda)$, and $\hat{\mathbf{H}}(\lambda)$ denote the Laplace transform of f, $\mathbf{g}$ and $\mathbf{H}$ with respect to time variable t. Thus, using Theorem 2.4 with the help of Theorem 2.2 and employing the same argumentation as in Sect.4 of Shibata [7], we have

Theorem 2.11. *Let $1 < p, q < \infty$, $N < r < \infty$, $\max(q, q') \leq r$ and $T > 0$. Let Ω be a bounded domain in $\mathbb{R}^N$, whose boundaires S and Γ are both $W_r^{2-1/r}$ compact hyper-surfaces. Then, for any initial data $(\theta_0, \mathbf{u}_0, \tau_0) \in W_q^1(\Omega) \times B_{q,p}^{2(1-1/p)}(\Omega)^N \times W_q^1(\Omega)^{N\times N}$ and right-hand sides f, $\mathbf{g}$, $\mathbf{H}$, and $\mathbf{k}$ satisfying (2.5) and (2.6), problem (2.17) admits a unique solution $(\theta, \mathbf{u}, \tau)$ with*

$$\theta \in W_p^1((0, T), W_q^1(\Omega)), \quad \mathbf{u} \in L_p((0, T), W_q^2(\Omega)^N) \cap W_p^1((0, T), L_q(\Omega)^N), \quad \tau \in W_p^1((0, T), W_q^1(\Omega)^{N\times N}).$$

Moreover, there exsits a positive constant η_2 such that θ, $\mathbf{u}$, and τ possess the estimate:

$$\begin{aligned} &\|e^{\eta s}\theta\|_{W_p^1((0,T),W_q^1(\Omega))} + \|e^{\eta s}\partial_s\mathbf{u}\|_{L_p((0,T),L_q(\Omega))} + \|e^{\eta s}\mathbf{u}\|_{L_p((0,T),W_q^2(\Omega))} + \|e^{\eta s}\tau\|_{W_p^1((0,T),W_q^1(\Omega))} \\ &\quad \leq C\Big\{\|\theta_0\|_{W_q^1(\Omega)} + \|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|\tau_0\|_{W_q^1(\Omega)} \\ &\qquad + \|e^{\eta s}(f, \mathbf{g}, \mathbf{H})\|_{L_p((0,T),W_q^{1,0}(\Omega))} + \|e^{\eta s}\mathbf{k}\|_{L_p((0,T),W_q^1(\Omega))} + \|e^{\eta s}\partial_s\mathbf{k}\|_{L_p((0,T),\mathbf{W}_q^{-1}(\Omega))}\Big\}\Big\} \end{aligned} \tag{2.18}$$

for any $\eta \in (0, \eta_2]$ with some positive constant C depending on η_2 but independent of T.

Under the above preparations, we finish proving Theorem 2.7. We look for a solution $(\theta, \mathbf{u}, \tau)$ of the form $\theta = \kappa + \omega$, $\mathbf{u} = \mathbf{v} + \mathbf{w}$ and $\tau = \psi + \varphi$ where $(\kappa, \mathbf{v}, \psi)$ and $(\omega, \mathbf{w}, \varphi)$ are solutions to the following problems :

$$\left\{\begin{array}{rl} \partial_t\kappa + \lambda_0\kappa + \rho_*\mathrm{div}\,\mathbf{v} = f & \text{in } \Omega \times (0,T), \\ \rho_*(\partial_t\mathbf{v} + \lambda_0\mathbf{v}) - \mathrm{Div}\,S(\mathbf{v}) + \nabla(P'(\rho_*)\kappa) - \beta\mathrm{Div}\,\psi = \mathbf{g} & \text{in } \Omega \times (0,T) \\ \partial_t\psi + \lambda_0\psi + \gamma\psi - \delta D(\mathbf{v}) = \mathbf{H} & \text{in } \Omega \times (0,T), \\ \mathbf{v} = 0 & \text{in } S \times (0,T), \\ (S(\mathbf{v}) - P'(\rho_*)\kappa\mathbf{I} + \beta\psi)\cdot\mathbf{n} = \mathbf{k} & \text{in } \Gamma \times (0,T), \\ (\kappa, \mathbf{v}, \psi)|_{t=0} = (\theta_0, \mathbf{u}_0, \tau_0) & \text{in } \Omega, \end{array}\right. \tag{2.19}$$

$$\left\{\begin{array}{rl} \partial_t\omega + \rho_*\mathrm{div}\,\mathbf{w} = \lambda_0\kappa & \text{in } \Omega \times (0,T), \\ \rho_*\partial_t\mathbf{w} - \mathrm{Div}\,S(\mathbf{w}) + \nabla(P'(\rho_*)\omega) - \beta\mathrm{Div}\,\varphi = \rho_*\lambda_0\mathbf{v} & \text{in } \Omega \times (0,T), \\ \partial_t\varphi + \gamma\varphi - \delta D(\mathbf{w}) = \lambda_0\psi & \text{in } \Omega \times (0,T), \\ \mathbf{w} = 0 & \text{in } S \times (0,T), \\ (S(\mathbf{w}) - P'(\rho_*)\omega\mathbf{I} + \beta\varphi)\mathbf{n} = 0 & \text{in } \Gamma \times (0,T), \\ (\omega, \mathbf{w}, \varphi)|_{t=0} = (0,0,0) & \text{in } \Omega, \end{array}\right. \tag{2.20}$$

respectively. By Theorem 2.11 we know the existence of κ, $\mathbf{v}$ and ψ that solve (2.19) and possess the estimate :

$$\begin{aligned} &\|e^{\eta s}\kappa\|_{W^1_p((0,T),W^1_q(\Omega))} + \|e^{\eta s}\partial_s\mathbf{v}\|_{L_p((0,T),L_q(\Omega))} + \|e^{\eta s}\mathbf{v}\|_{L_p((0,T),W^2_q(\Omega))} + \|e^{\eta s}\psi\|_{W^1_p((0,T),W^1_q(\Omega))} \\ &\leq C\Big\{\|\theta_0\|_{W^1_q(\Omega)} + \|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|\tau_0\|_{W^1_q(\Omega)} \\ &\quad + \|e^{\eta s}(f,\mathbf{H},\mathbf{k})\|_{L_p((0,T),W^1_q(\Omega))} + \|e^{\eta s}\mathbf{g}\|_{L_p((0,T),L_p(\Omega))} + \|e^{\eta s}\partial_s\mathbf{k}\|_{L_p((0,T),\mathbf{W}^{-1}_q(\mathbb{R}^N))}\Big\}. \end{aligned} \tag{2.21}$$

For the sake of simplicity, we set

$$\begin{aligned} \mathbf{J}_{p,q} &= \|\theta_0\|_{W^1_q(\Omega)} + \|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|\tau_0\|_{W^1_q(\Omega)} \\ &\quad + \|e^{\eta s}(f,\mathbf{H},\mathbf{k})\|_{L_p((0,T),W^1_q(\Omega))} + \|e^{\eta s}\mathbf{g}\|_{L_p((0,T),L_p(\Omega))} + \|e^{\eta s}\partial_s\mathbf{k}\|_{L_p((0,T),\mathbf{W}^{-1}_q(\mathbb{R}^N))} \end{aligned}$$

where $\eta = \min(\eta_1, \eta_2)/2$, and η_1 and η_2 are the positive numbers appearing in Corollary 2.10 and Theorem 2.11, respectively. Let $\{T(t)\}_{t\geq 0}$ be the semigroup associated with (2.12) and let $\mathbf{z}(x,s) = \mathbf{v}(x,s) - d(S)\sum_{\ell=1}^M(\mathbf{v}(\cdot,s),\mathbf{p}_\ell)_\Omega\mathbf{p}_\ell$. Defining $\tilde\omega$, $\tilde{\mathbf{w}}$ and $\tilde\varphi$ by

$$(\tilde\omega(\cdot,t), \tilde{\mathbf{w}}(\cdot,t), \tilde\varphi(\cdot,t)) = \lambda_0\int_0^t \mathbf{T}(t-s)(\kappa(\cdot,s), \rho_*\mathbf{z}(\cdot,s), \psi(\cdot,s))ds \tag{2.22}$$

by the Duhamel principle we see that $\tilde\omega$, $\tilde{\mathbf{w}}$ and $\tilde\varphi$ satisfy the equations

$$\left\{\begin{array}{rl} \partial_t\tilde\omega + \rho_*\mathrm{div}\,\tilde{\mathbf{w}} = \lambda_0\kappa & \text{in } \Omega \times (0,T), \\ \rho_*\partial_t\tilde{\mathbf{w}} - \mathrm{Div}\,S(\tilde{\mathbf{w}}) + \nabla(P'(\rho_*)\tilde\omega) - \beta\mathrm{Div}\,\tilde\varphi = \rho_*\lambda_0(\mathbf{v} - d(S)\sum_{\ell=1}^M(\mathbf{v}(\cdot,s),\mathbf{p}_\ell)_\Omega\mathbf{p}_\ell) & \text{in } \Omega \times (0,T), \\ \partial_t\tilde\varphi + \gamma\tilde\varphi - \delta D(\tilde{\mathbf{w}}) = \lambda_0\psi & \text{in } \Omega \times (0,T), \\ \tilde{\mathbf{w}} = 0 & \text{in } S \times (0,T), \\ (S(\tilde{\mathbf{w}}) - P'(\rho_*)\tilde\omega\mathbf{I} + \beta\tilde\varphi)\cdot\mathbf{n} = 0 & \text{in } \Gamma \times (0,T), \\ (\tilde\omega, \tilde{\mathbf{w}}, \tilde\varphi)|_{t=0} = (0,0,0) & \text{in } \Omega. \end{array}\right. \tag{2.23}$$

Since $(\mathbf{z}(\cdot,s),\mathbf{p}_\ell)_\Omega = 0$ for any $\ell = 1,\dots,M$ and $s \in (0,T)$ when $S = \emptyset$; by Corollary 2.10 we have

$$\|(\tilde\omega(\cdot,t), \tilde{\mathbf{w}}(\cdot,t), \tilde\varphi(\cdot,t))\|_{W^{1,0}_q(\Omega)} \leq C\int_0^t e^{-\eta_1(t-s)}\|(\kappa(\cdot,s), \mathbf{z}(\cdot,s), \psi(\cdot,s))\|_{W^{1,0}_q(\Omega)}\,ds.$$

Thus, by Hölder's inequality and the change of the integral order, we have

$$\int_0^T (e^{\eta t}\|(\tilde\omega(\cdot,t),\tilde{\mathbf{w}}(\cdot,t),\tilde\varphi(\cdot,t))\|_{W^{1,0}_q(\Omega)})^p\,dt \le C\eta^{-p}\int_0^T (e^{\eta s}\|(\kappa(\cdot,s),\mathbf{z}(\cdot,s),\psi(\cdot,s))\|_{W^{1,0}_q(\Omega)})^p\,ds,$$

which, combined with (2.21), furnishes that

$$\|e^{\eta s}(\tilde\omega,\tilde{\mathbf{w}},\tilde\varphi)\|_{L_p((0,T),W^{1,0}_q(\Omega)} \le C\mathbf{J}_{p,q}. \tag{2.24}$$

Since $\tilde\omega$, $\tilde\varphi$ and $\tilde{\mathbf{w}}$ satisfy the shifted equations:

$$\begin{cases} \tilde\omega_t + \lambda_0\tilde\omega + \rho_*\mathrm{div}\,\tilde{\mathbf{w}} = \lambda_0(\tilde\omega+\kappa) & \text{in } \Omega\times(0,T),\\ \rho_*(\tilde{\mathbf{w}}_t+\lambda_0\tilde{\mathbf{w}}) - \mathrm{Div}\,\mathbf{S}(\tilde{\mathbf{w}}) + \nabla(P'(\rho_*)\tilde\omega) - \beta\mathrm{Div}\,\tilde\varphi & \\ \quad = \rho_*\lambda_0(\tilde{\mathbf{w}}+\mathbf{v} - d(S)\sum_{\ell=1}^M(\mathbf{v}(\cdot,s),\mathbf{p}_\ell)_\Omega\mathbf{p}_\ell) & \text{in } \Omega\times(0,T),\\ \partial_t\tilde\varphi + \lambda_0\tilde\varphi + \gamma\tilde\varphi - \delta\mathbf{D}(\tilde{\mathbf{w}}) = \lambda_0(\tilde\varphi+\psi) & \text{in } \Omega\times(0,T),\\ \tilde{\mathbf{w}} = 0 & \text{in } S\times(0,T),\\ (S(\tilde{\mathbf{w}}) - P'(\rho_*)\tilde\omega\mathbf{I} + \beta\tilde\varphi)\cdot\mathbf{n} = 0 & \text{in } \Gamma\times(0,T),\\ (\tilde\omega,\tilde{\mathbf{w}},\tilde\varphi)|_{t=0} = (0,0,0) & \text{in } \Omega, \end{cases} \tag{2.25}$$

by Theorem 2.11, (2.21) and (2.24) we have

$$\|e^{\eta s}\tilde\omega\|_{W^1_p((0,T),W^1_q(\Omega))} + \|e^{\eta s}\partial_s\tilde{\mathbf{w}}\|_{L_p((0,T),L_q(\Omega))} + \|e^{\eta s}\tilde{\mathbf{w}}\|_{L_p((0,T),W^2_q(\Omega))} + \|e^{\eta s}\tilde\varphi\|_{W^1_p((0,T),W^1_q(\Omega))} \le C\mathbf{J}_{p,q}. \tag{2.26}$$

When $S\neq\emptyset$, setting $\omega=\tilde\omega$, $\varphi=\tilde\varphi$ and $\mathbf{w}=\tilde{\mathbf{w}}$, we have Theorem 2.7.

Finally, we consider the case $S=\emptyset$. Let

$$\omega=\tilde\omega,\ \ \varphi=\tilde\varphi,\ \ \mathbf{w} = \tilde{\mathbf{w}} + \lambda_0\rho_* d(S)\sum_{\ell=1}^M\int_0^t(\mathbf{v}(\cdot,s),\mathbf{p}_\ell)_\Omega ds\,\mathbf{p}_\ell.$$

Since (2.11)holds and $\mathbf{p}_\ell$ is the first order polynomial, we have $\mathrm{div}\,\mathbf{u} = \mathrm{div}\,\mathbf{v} + \mathrm{div}\,\tilde{\mathbf{w}}$, $\mathbf{S}(\mathbf{u}) = \mathbf{S}(\mathbf{v})+\mathbf{S}(\tilde{\mathbf{w}})$, $\mathbf{D}(\mathbf{u}) = \mathbf{D}(\mathbf{v})+\mathbf{D}(\tilde{\mathbf{w}})$, and $\nabla^2\mathbf{u} = \nabla^2(\mathbf{v}+\tilde{\mathbf{w}})$. Thus, by (2.22) and (2.25) we see that θ, $\mathbf{u}$ and τ satisfy the equations (2.1). Moreover, by (2.21) and (2.26), we have

$$\|e^{\eta s}\theta\|_{W^1_p((0,T),W^1_q(\Omega))} + \|e^{\eta s}\partial_s\mathbf{u}\|_{L_p((0,T),L_q(\Omega))} + \|e^{\eta s}\mathbf{D}(\mathbf{u})\|_{L_p((0,T),L_q(\Omega))} + \|e^{\eta s}\nabla^2\mathbf{u}\|_{L_p((0,T),L_q(\Omega))} + \|e^{\eta s}\tau\|_{W^1_p((0,T),W^1_q(\Omega))} \le C\mathbf{J}_{p,q}. \tag{2.27}$$

Using the first Korn inequality, we have

$$\|\mathbf{u}(\cdot,s)\|_{W^1_q(\Omega)} \le C\{\|\mathbf{D}(\mathbf{u}(\cdot,s))\|_{L_q(\Omega)} + \sum_{\ell=1}^M|(\mathbf{u}(\cdot,s),\mathbf{p}_\ell)_\Omega|\},$$

which, combined with (2.27), furnishes that

$$\begin{aligned}\|e^{\eta s}\mathbf{u}(\cdot,s)\|_{L_p((0,t),W^1_q(\Omega))} &\le C\{\|e^{\eta s}\mathbf{D}(\mathbf{u}(\cdot,s))\|_{L_p((0,t),L_q(\Omega)} + \sum_{\ell=1}^M\Big(\int_0^t(e^{\eta s}|(\mathbf{u}(\cdot,s),\mathbf{p}_\ell)_\Omega|)^pds\Big)^{1/p}\}\\ &\le C\{\mathbf{J}_{p,q} + \sum_{\ell=1}^M\Big(\int_0^t(e^{\eta s}|(\mathbf{u}(\cdot,s),\mathbf{p}_\ell)_\Omega|)^pds\Big)^{1/p}\}\end{aligned} \tag{2.28}$$

Thus, combining (2.27) and (2.28), we have

$$\|e^{\eta s}\theta\|_{W^1_p((0,T),W^1_q(\Omega))} + \|e^{\eta s}\partial_s\mathbf{u}\|_{L_p((0,T),L_q(\Omega))} + \|e^{\eta s}\mathbf{u}\|_{L_p((0,T),W^2_q(\Omega))} + \|e^{\eta s}\tau\|_{W^1_p((0,T),W^1_q(\Omega))} \le C\{\mathbf{J}_{p,q} + \sum_{\ell=1}^M\Big(\int_0^t(e^{\eta s}|(\mathbf{u}(\cdot,s),\mathbf{p}_\ell)_\Omega|)^pds\Big)^{1/p}\}.$$

This completes the proof of Theorem 2.7.

3 A proof of Theorem 2.9

In this section, we prove Theorem 2.9. For this purpose, first we consider problem (2.2) with $\lambda = 0$, that is

$$\begin{cases} \rho_* \mathrm{div}\, \mathbf{u} = f & \text{in } \Omega, \\ -\mathrm{Div}\, \mathbf{S}(\mathbf{u}) + P'(\rho_*)\nabla\theta - \beta \mathrm{Div}\, \tau = \mathbf{g} & \text{in } \Omega, \\ \gamma\tau - \delta \mathbf{D}(\mathbf{u}) = \mathbf{H} & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } S, \\ (\mathbf{S}(\mathbf{u}) - P'(\rho_*)\theta \mathbf{I} + \beta\tau)\mathbf{n} = \mathbf{k} & \text{on } \Gamma. \end{cases} \tag{3.1}$$

With the help of the following Lemma, we start to prove Theorem 2.9.

Lemma 3.1. *Let $1 < q < \infty$, $N < r < \infty$ and $\lambda_1 > 0$. Assume that $max(q, q') \leq r$. Let ℓ_b be the number defined in Theorem 1.1. Let Ω be a bounded domain in $\mathbb{R}^N$, whose boundaries S and Γ are both $W_r^{\ell_b - 1/r}$ compact hyper-surfaces. Then, for any $(f, \mathbf{g}, \mathbf{H}) \in W_q^{1,0}(\Omega)$ and $\mathbf{k} \in W_q^1(\Omega)^N$ satisfying the condition:*

$$(\mathbf{g}, \mathbf{p}_\ell)_\Omega + (\mathbf{k}, \mathbf{p}_\ell)_\Gamma = 0 \quad (\ell = 1, \ldots, M) \tag{3.2}$$

when $S = \emptyset$, problem (3.1) admits unique solutions $\theta \in W_q^1(\Omega)$ and $\mathbf{u} \in \dot{W}_q^2(\Omega)$ possessing the estimate:

$$\|(\theta, \mathbf{u}, \tau)\|_{W_q^{1,2}(\Omega)} \leq C(\|(f, \mathbf{g}, \mathbf{H})\|_{W_q^{1,0}(\Omega)} + \|\mathbf{k}\|_{W_q^1(\Omega)}). \tag{3.3}$$

Remark 3.2. Recall that $\dot{W}_q^2(\Omega)^N$ is the set of all $\mathbf{u} \in W_q^2(\Omega)^N$ which satisfies (2.10).

Proof. The technical proof of the Lemma can be seen in [4]. □

In the sequel, we prove Theorem 2.9. In view of Lemma 3.1 by the small perturbation argument, there exists a small $\lambda_0 > 0$ such that problem (2.2) can be solved with $\lambda \in \mathbb{C}$ and $|\lambda| \leq \lambda_0$. Namely, Theorem 2.9 holds for $\lambda \in \mathbb{C}$ with $|\lambda| \leq \lambda_0$. Furthermore, we consider the case where $\mathrm{Re}\,\lambda \geq 0$ and $\lambda_0 \leq |\lambda| \leq \lambda_1$. In this case, setting $\theta = \lambda^{-1}(f - \rho_* \mathrm{div}\, \mathbf{u})$ and $\tau = (\lambda + \gamma)^{-1}(\delta \mathbf{D}(\mathbf{u}) + \mathbf{H})$ in (2.2), we have a generalized Lamé system:

$$\rho_* \lambda \mathbf{u} - \mathrm{Div}\, \mathbf{S}_\lambda(\mathbf{u}) = \mathbf{g}' \ \text{ in } \Omega, \quad \mathbf{u} = 0 \ \text{ on } S, \quad \mathbf{S}_\lambda(\mathbf{u})\mathbf{n} = \mathbf{k}' \ \text{ on } \Gamma, \tag{3.4}$$

where we have set

$$\begin{aligned} \mathbf{S}_\lambda(\mathbf{u}) &= (\mu + \beta(\lambda + \gamma)^{-1}\delta)\mathbf{D}(\mathbf{u}) + ((\nu - \mu) + P'(\rho_*)\rho_*\lambda^{-1})\mathrm{div}\, \mathbf{u}, \\ \mathbf{g}' &= \mathbf{g} - (P'(\rho_*)\lambda^{-1}\nabla f - \beta(\lambda + \gamma)^{-1}\mathrm{Div}\, \mathbf{H}), \\ \mathbf{k}' &= \mathbf{k} + (P'(\rho_*)\lambda^{-1} f \mathbf{I} - \beta(\lambda + \gamma)^{-1}\mathbf{H})\mathbf{n}. \end{aligned}$$

Since $\lambda_0 \leq |\lambda| \leq \lambda_1$, by $\|h\mathbf{n}\|_{W_q^i(\Omega)} \leq C\|h\|_{W_q^i(\Omega)}$ $(i = 0, 1)$, we have

$$\|\mathbf{g}'\|_{L_q(\Omega)} + \|\mathbf{k}'\|_{W_q^1(\Omega)} \leq C_{\lambda_0, \lambda_1}(\|(f, \mathbf{g}, \mathbf{H})\|_{W_q^{1,0}(\Omega)} + \|\mathbf{k}\|_{W_q^1(\Omega)}).$$

To solve (3.4), first for fixed λ we consider the equations:

$$\rho_* \kappa \mathbf{u} - \mathrm{Div}\, \mathbf{S}_\lambda(\mathbf{u}) = \mathbf{g}' \ \text{ in } \Omega, \quad \mathbf{u} = 0 \ \text{ on } S, \quad \mathbf{S}_\lambda(\mathbf{u})\mathbf{n} = \mathbf{k}' \ \text{ on } \Gamma, \tag{3.5}$$

with new resolvent parameter $\kappa \in \mathbb{R}$. Note that if $(\mathbf{g}, \mathbf{k})$ satisfies (3.2), then $(\mathbf{g}', \mathbf{k}')$ also satisfies (3.2). Employing the same argumentation as in Shibata and Tanaka [9] or Enomoto, von Below and Shibata [2], we see that there exists a large $\kappa_0 > 0$ depending on λ such that for any $\kappa \geq \kappa_0$ and $(\mathbf{g}', \mathbf{k}') \in L_q(\Omega)^N \times W_q^1(\Omega)^N$ satisfying (3.2) problem (3.5) admits a unique solution $\mathbf{u} \in \dot{W}_q^2(\Omega)^N$. Since the solution operator of problem (3.5) with $\kappa = \kappa_0$ is compact, by the Riesz-Schauder theory we see that the uniqueness implies the existence in problem (3.4). Thus, we examine the uniqueness. Let $\mathbf{u} \in \dot{W}_q^2(\Omega)^N$ be a solution of the homogeneous equation:

$$\rho_* \lambda \mathbf{u} - \mathrm{Div}\, \mathbf{S}_\lambda(\mathbf{u}) = 0 \ \text{ in } \Omega, \quad \mathbf{u} = 0 \ \text{ on } S, \quad \mathbf{S}_\lambda(\mathbf{u})\mathbf{n} = 0 \ \text{ on } \Gamma, \tag{3.6}$$

First we consider the case $2 \leq q < \infty$. In this case, $\mathbf{u} \in \dot{W}_2^2(\Omega)^N$. Thus, multiplying the first equation (3.6) by $\mathbf{u}$ and using the divergence theorem of Gauß, we have

$$0 = \rho_*\lambda\|\mathbf{u}\|_{L_2(\Omega)}^2 + \frac{1}{2}(\mu + \beta(\lambda+\gamma)^{-1}\delta)\|\mathbf{D}(\mathbf{u})\|_{L_2(\Omega)}^2 + ((\nu-\mu) + P'(\rho_*)\rho_*\lambda^{-1})\|\operatorname{div}\mathbf{u}\|_{L_2(\Omega)}^2. \tag{3.7}$$

When $\operatorname{Re}\lambda \geq 0$, $\operatorname{Re}\rho_*\lambda^{-1} \geq 0$ and $\operatorname{Re}\beta(\lambda+\gamma)^{-1}\delta \geq 0$, so that taking the real part of (3.7), we have

$$0 \geq \rho_*\operatorname{Re}\lambda\|\mathbf{u}\|_{L_2(\Omega)}^2 + \frac{\mu}{2}\|\mathbf{D}(\mathbf{u})\|_{L_2(\Omega)}^2 + (\nu-\mu)\|\operatorname{div}\mathbf{u}\|_{L_2(\Omega)}^2. \tag{3.8}$$

Since $\|\operatorname{div}\mathbf{u}\|_{L_2(\Omega)}^2 \leq (N/4)\|\mathbf{D}(\mathbf{u})\|_{L_2(\Omega)}^2$, by (3.8) we have

$$0 \geq \Big(\nu - \frac{N-2}{N}\mu\Big)\|\operatorname{div}\mathbf{u}\|_{L_2(\Omega)}^2$$

provided that $\operatorname{Re}\lambda \geq 0$. Since we assume that $\nu - \frac{N-2}{N}\mu > 0$, we have $\operatorname{div}\mathbf{u} = 0$, so that by (3.8) and the assumption that $\mu > 0$, we have $\mathbf{D}(\mathbf{u}) = 0$ provided that $\operatorname{Re}\lambda \geq 0$. When $S \neq \emptyset$, we have $\mathbf{u}|_S = 0$, so that the first Korn inequality: $\|\nabla\mathbf{u}\|_{L_2(\Omega)} \leq C\|\mathbf{D}(\mathbf{u})\|_{L_2(\Omega)}$ does hold. Therefore, $\nabla\mathbf{u} = 0$, which implies that $\mathbf{u}$ is constant. But, $\mathbf{u}|_S = 0$, so that finally we arrive at $\mathbf{u} = 0$. On the other hand, when $S = \emptyset$, $\mathbf{u}$ satisfies (2.10), so that $\mathbf{u} = 0$ too. Therefore, we have the uniqueness, which implies the unique existence of solutions to problem (3.4) for each λ with $\lambda_0 \leq |\lambda| \leq \lambda_1$ when $2 \leq q < \infty$. When $1 < q < 2$, the uniqueness follows from the existence for the dual problem, so that in this case we also have the unique existence of solutions. If we know the unique exstence of solutions to (3.4) for one λ_2, by the small perturbation argument there exists a small number δ depending on λ_2 such that the unique exstence of solutions to (3.4) holds for $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_2| \leq \delta$. Since the set $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq 0,\ \lambda_0 \leq |\lambda| \leq \lambda_1\}$ is compact, we have the unique existence theorem holds for any $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq 0,\ \lambda_0 \leq |\lambda| \leq \lambda_1\}$ with uniform constant C in the estimate (3.3). This completes the proof of Theorem 2.9.

4 A proof of Theorem 1.3

To prove Theorem 1.3, we start with

Lemma 4.1. *Let $1 < p, q < \infty$, let T be any positive number and let Ω be a bounded domain in $\mathbb{R}^N$, whose boundary Γ is a $W_r^{2-1/r}$ compact hyper-surface with $N < r < \infty$. Then, the following two assertions hold:*

(1) *We have*

$$\sup_{t \in (0,T)} \|u(t)\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \leq C\{\|u(\cdot,0)\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \mathbf{I}_u(T)\}$$

for any $u \in L_p((0,T), W_q^2(\Omega)) \cap W_p^1((0,T), L_q(\Omega))$ with some constant C independent of T. Here, we have set

$$\mathbf{I}_u(T) = \|\partial_t u\|_{L_p((0,T),L_q(\Omega))} + \|u\|_{L_p((0,T),W_q^2(\Omega))}$$

(2) *Assume that* $\max(q, q') \leq r$. *Then, we have*

$$\begin{aligned}
&\|\nabla u\|_{\mathbf{W}_q^{-1}(\Omega)} \leq C\|u\|_{L_q(\Omega)} && \text{for any } u \in L_q(\Omega),\\
&\|uv\|_{\mathbf{W}_q^{-1}(\Omega)} \leq C\|u\|_{\mathbf{W}_q^{-1}(\Omega)}\|v\|_{W_q^1(\Omega)} && \text{for any } u \in \mathbf{W}_q^{-1}(\Omega),\ v \in W_q^1(\Omega),\\
&\|uv\|_{W_q^{-1}(\Omega)} \leq C\|u\|_{L_q(\Omega)}\|v\|_{L_q(\Omega)} && \text{for any } u,\ v \in L_q(\Omega).
\end{aligned} \tag{4.1}$$

Proof. Lemma has been proved in [3] (cf. also in [6]), so that we may omit the proof. □

From now on, we prove Theorem 1.3. Let ϵ be a small positive number and we assume that initial data $(\theta_0, \mathbf{v}_0, \tau_0) \in \mathcal{D}_{q,p}(\Omega)$ satisfies the conditions:

$$\begin{gathered}
\frac{2}{3}\rho_* < \rho_* + \theta_0 < \frac{4}{3}\rho_*,\\
\|\theta_0\|_{W_q^1(\Omega)} + \|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|\tau_0\|_{W_q^1(\Omega)} \leq \epsilon
\end{gathered} \tag{4.2}$$

and the orthogonal condition (1.18). Since we choose an ϵ small enough eventually, we may assume that $0 < \epsilon \leq 1$. Thus, by Theorem 1.1, there exists a $T_0 > 0$ such that problem (1.10) admits a unique solution with $T = T_0$. Let T be a positive number and we assume that problem (1.10) admits a solution $(\theta, \mathbf{u}, \omega)$ with

$$\theta \in W^1_p((0,T), W^1_q(\Omega)),\ \ \mathbf{u} \in L_p((0,T), W^2_q(\Omega)^N) \cap W^1_p((0,\infty), L_q(\Omega)^N),\ \ \omega \in W^1_p((0,T), W^1_q(\Omega)^{N\times N})$$

satisfying the condition:

$$\frac{1}{3}\rho_* < \rho_* + \theta(x,t) < \frac{5}{3}\rho_* \quad \text{for any } (x,t) \in \Omega \times (0,T), \qquad \sup_{0<t<T} \|\int_0^t \nabla \mathbf{u}(\cdot,s)\,ds\|_{L_\infty(\Omega)} \leq \sigma. \tag{4.3}$$

where σ is the positive number appearing in (1.7). We may assume that $0 < \sigma \leq 1$ and $T \geq T_0$. Let

$$\mathbf{I}(t) = \|e^{\eta s}\theta\|_{W^1_p((0,t),W^1_q(\Omega))} + \|e^{\eta s}\partial_s \mathbf{u}\|_{L_p((0,t),L_q(\Omega))} + \|e^{\eta s}\mathbf{u}\|_{L_p((0,t),W^2_q(\Omega))} + \|e^{\eta s}\omega\|_{W^1_p((0,t),W^1_q(\Omega))}$$

with some positive constant η for which Theorem 2.7 holds. Our main task is to prove

$$\mathbf{I}(t) \leq M_1(\epsilon + \mathbf{I}(t)^2) \tag{4.4}$$

with some constant M_1 independent of ϵ and T. To prove (4.4), we start with

$$\begin{aligned} \|\theta(\cdot,t)\|_{W^1_q(\Omega)} &\leq C(\|\theta_0\|_{W^1_q(\Omega)} + \mathbf{I}(t)), \\ \|\mathbf{u}(\cdot,t)\|_{B^{2(1-1/p)}_{q,p}(\Omega)} &\leq C(\|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \mathbf{I}(t)), \\ \|\omega(\cdot,t)\|_{W^1_q(\Omega)} &\leq C(\|\tau_0\|_{W^1_q(\Omega)} + \mathbf{I}(t)) \end{aligned} \tag{4.5}$$

In fact, writing $\theta(x,t) = \theta_0 + \int_0^t \partial_s\theta(\cdot,s)ds$ and $\omega(x,t) = \theta_0 + \int_0^t \partial_s\omega(\cdot,s)ds$, we have the first and third inequality in (4.5). The second inequality in (4.5) follows from Lemma 4.1 (1). Hereinafter, the letter C stands for generic constants independent of T and ϵ. Its value may differ even in a single chain of inequalities. By Hölder's inequality, we have

$$\int_0^t \|\mathbf{u}(\cdot,s)\|_{W^2_q(\Omega)}ds \leq C\Big(\int_0^t e^{-p'\gamma s}ds\Big)^{1/p'}\Big(\int_0^t (e^{\gamma s}\|\mathbf{u}(\cdot,s)\|_{W^2_q(\Omega)})^p ds\Big)^{1/p} \leq C\mathbf{I}(t). \tag{4.6}$$

To estimate the products, we use the Sobolev embedding theorem:

$$\|\prod_{j=1}^m f_j\|_{W^1_q(\Omega)} \leq C\prod_{j=1}^m \|f_j\|_{W^1_q(\Omega)}, \quad \|f\|_{L_\infty(\Omega)} \leq C\|f\|_{W^1_q(\Omega)} \tag{4.7}$$

because $N < q < \infty$. Since $2 < p < \infty$, we have $B^{2(1-1/p)(\Omega)}_{q,p} \subset W^1_q(\Omega)$, that is

$$\|f\|_{W^1_q(\Omega)} \leq C\|f\|_{B^{2(1-1/p)}_{q,p}(\Omega)}. \tag{4.8}$$

Recall the definition of nonlinear terms $f(\theta,\mathbf{u},\omega)$, $\mathbf{g}(\theta,\mathbf{u},\omega)$, $\mathbf{h}(\theta,\mathbf{u},\omega)$ and $\mathbf{L}(\theta,\mathbf{u},\omega)$. Using $\|h\mathbf{n}\|_{W^i_q(\Omega)} \leq C\|h\|_{W^i_q(\Omega)}$ $(i = 0,1)$, (4.3), (4.5), (4.6), (4.7), and (4.8) and noting that $V_{\rm div}(0) = 0$, $V_D(0) = 0$, $V_{\rm Div}(0)$ and $V_0(0) = 0$, we have

$$\|e^{\gamma s}(f(\theta,\mathbf{u},\omega), \mathbf{h}(\theta,\mathbf{u},\omega), \mathbf{L}(\theta,\mathbf{u},\omega))\|_{L_p((0,t),W^1_q(\Omega))} + \|e^{\gamma s}\mathbf{g}(\theta,\mathbf{u},\omega)\|_{L_p((0,t),L_q(\Omega))} \leq C(\epsilon + \mathbf{I}(t)^2). \tag{4.9}$$

By (**??**), (4.1), (4.3), (4.5), (4.6), (4.7), and (4.8) and noting that $V_{\rm div}(0) = 0$, $V_D(0) = 0$, $V_{\rm Div}(0)$ and $V_0(0) = 0$, we also have

$$\|e^{\gamma s}\partial_s \mathbf{h}(\theta,\mathbf{u},\omega)]\|_{L_p((0,t),\mathbf{W}^{-1}_q(\mathbb{R}^N))} \leq C(\epsilon + \mathbf{I}(t)^2). \tag{4.10}$$

To obtain (4.9) and (4.10), we used the fact that $(\epsilon + \mathbf{I}(t))\mathbf{I}(t) \leq (1/2)\epsilon^2 + (3/2)\mathbf{I}(t)^2 \leq 2(\epsilon + \mathbf{I}(t)^2)$, because of $0 < \epsilon \leq 1$.

Applying Theorem 2.7 to problem (1.10) and using (4.9) and (4.10), we have

$$\mathbf{I}(t) \le C\{\epsilon + \mathbf{I}(t)^2 + d(S)\sum_{\ell=1}^{M}\Big(\int_0^t (e^{\gamma s}|(\mathbf{u}(\cdot,s),\mathbf{p}_\ell)_\Omega|)^p ds\Big)^{1/p}\}. \tag{4.11}$$

Now, we consider the case where $S = \emptyset$, namely $d(S) = 1$. According to the argumentation due to G. Ströhmer [10], the Lagrange transform $x = \mathbf{X}_{\mathbf{u}}(\xi,t) = \xi + \int_0^t \mathbf{u}(\xi,s)\,ds$ is a bijection from Ω onto $\Omega_t = \{x = \mathbf{X}_{\mathbf{u}}(\xi,t) \mid \xi \in \Omega\}$ and from Γ onto $\Gamma_t = \{x = \mathbf{X}_{\mathbf{u}}(\xi,t) \mid \xi \in S\}$, so that denoting the inverse map by $\mathbf{Y}(x,t)$, by (1.9) we see that $\rho(x,t) = \rho_* + \theta(\mathbf{Y}(x,t),t)$, $\mathbf{v}(x,t) = \mathbf{u}(\mathbf{Y}(x,t),t)$, and $\tau(x,t) = \omega(\mathbf{Y}(x,t),t)$ satisfy the equations (1.1). Since we assume that $\tau_0 \in Sym(\mathbb{R}^N)$, we know that $\tau \in Sym(\mathbb{R}^N)$, too. Let J be the determinant of the Jacobi matrix of the transformation: $x = \mathbf{X}_{\mathbf{u}}(\xi,t)$, and then noting that $\rho(\xi + \int_0^t \mathbf{u}(\xi,s),t) = \rho_* + \theta(\xi,t)$ and $\mathbf{v}(\xi + \int_0^t \mathbf{u}(\xi,s),t) = \mathbf{u}(\xi,t)$ we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega_t}(\rho(\cdot,t)\mathbf{v}(\cdot,t),\mathbf{p}_\ell)dx \\
&\quad = \int_\Omega \partial_t\Big[(\rho_* + \theta(\xi,t))\mathbf{u}(\xi,t)\Big]\cdot \mathbf{p}_\ell(\xi + \int_0^t \mathbf{u}(\xi,s)ds)J(\xi,t)\,d\xi \\
&\quad + \int_\Omega (\rho_* + \theta(\xi,t))\mathbf{u}(\xi,t)\cdot\partial_t\Big[\mathbf{p}_\ell(\xi + \int_0^t \mathbf{u}(\xi,s)ds)\Big]J(\xi,t)\,d\xi \\
&\quad + \int_\Omega (\rho_* + \theta(\xi,t))\mathbf{u}(\xi,t)\cdot \mathbf{p}_\ell(\xi + \int_0^t \mathbf{u}(\xi,s)ds)\partial_t J(\xi,t)\,d\xi.
\end{aligned}$$

Since $\partial_t J(\xi,t) = (\operatorname{div}\mathbf{v}(x,t))J(\xi,t)$, by (1.1) we have

$$\begin{aligned}
&\partial_t((\rho_* + \theta(\xi,t))\mathbf{u}(\xi,t))J(\xi,t) + (\rho_* + \theta(\xi,t))\mathbf{u}(\xi,t)\partial_t J(\xi,t) \\
&\quad = (\operatorname{Div}\mathbf{T}(\mathbf{v},\rho) + \beta\operatorname{Div}\tau)J(\xi,t).
\end{aligned}$$

Moreover, representing $\mathbf{p}_\ell(x) = (\sum_{j=1}^N a_{ij}x_j,\ldots,\sum_{j=1}^N a_{Nj}x_j) + \mathbf{b}$ with $a_{ij} + a_{ji} = 0$. we have

$$\mathbf{u}(\xi,t)\cdot\partial_t(\mathbf{p}_\ell(\xi + \int_0^t \mathbf{u}(\xi,s)\,ds)) = \sum_{i,j=1}^N a_{ij}u_i(\xi,t)u_j(\xi,t) = \frac{1}{2}\sum_{i,j=1}^N (a_{ij} + a_{ji})u_i(\xi,t)u_j(\xi,t) = 0.$$

Summing up these two facts and using the symmetry of τ and (2.11), we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega_t}(\rho(\cdot,t)\mathbf{v}(\cdot,t),\mathbf{p}_\ell)dx = (\operatorname{Div}\mathbf{T}(\mathbf{v},\rho) + \beta\operatorname{Div}\tau,\mathbf{p}_\ell)_{\Omega_t} \\
&\quad = -\frac{\mu}{2}(\mathbf{D}(\mathbf{v}),\mathbf{D}(\mathbf{p}_\ell))_{\Omega_t} - (\nu-\mu)(\operatorname{div}\mathbf{v},\operatorname{div}\mathbf{p}_\ell)_{\Omega_t} + (P(\rho),\operatorname{div}\mathbf{p}_\ell)_{\Omega_t} - \frac{\beta}{2}(\tau,\mathbf{D}(\mathbf{p}_\ell))_{\Omega_t} = 0.
\end{aligned}$$

Thus,

$$\int_\Omega (\rho_* + \theta(\xi,t))\mathbf{u}(\xi,t)\mathbf{p}_\ell(\xi + \int_0^t \mathbf{u}(\xi,s)ds)J(\xi,t)d\xi = ((\rho_* + \theta_0)\mathbf{v}_0,\mathbf{p}_\ell)_\Omega = 0 \quad (\ell = 1,\ldots,M) \tag{4.12}$$

for any $t \in (0,T)$. Since $J(\xi,t) = \det(\mathbf{I} + \mathbf{V}_0(\int_0^t \nabla\mathbf{u}(\xi,s)ds))$ and $\mathbf{V}_0(0) = 0$, we may write $J(\xi,t)$ in the form:

$$J(\xi,t) = 1 + v_0(\int_0^t \nabla\mathbf{u}(\xi,s)\,ds)$$

where $v_0 = v_0(\mathbf{K})$ is a C^∞ function with respect to $\mathbf{K}$ defined on $|\mathbf{K}| \le \sigma$ with $v_0(0) = 0$. Moreover, we write

$$\mathbf{p}_\ell(\xi + \int_0^t \mathbf{u}(\xi,s)\,ds) = \mathbf{p}_\ell(\xi) + A_\ell\int_0^t \mathbf{u}(\xi,s)\,ds$$

with some constant matrix A_ℓ. And then, by (4.12) we have

$$
\begin{aligned}
(\mathbf{u}(\cdot,t),\mathbf{p}_\ell)_\Omega &= -\rho_*^{-1}\Big(\rho_*(\mathbf{u}(\cdot,t),\mathbf{p}_\ell v_0(\int_0^t \nabla\mathbf{u}(\xi,s)ds))_\Omega \\
&+\rho_*(\mathbf{u}(\cdot,t),A_\ell\int_0^t \mathbf{u}(\cdot,s)\,dsJ(\cdot,t))_\Omega+\int_\Omega \theta(\xi,t)\mathbf{u}(\xi,t)\mathbf{p}_\ell(\xi+\int_0^t \mathbf{u}(\xi,s)\,ds)J(\xi,t)\,d\xi\Big).
\end{aligned}
\tag{4.13}
$$

Thus, using (4.3) and (4.13) we have

$$|(\mathbf{u}(\cdot,t),\mathbf{p}_\ell)_\Omega| \le C(\|\theta_0\|_{W_q^1(\Omega)}+\mathbf{I}(t))\|\mathbf{u}(\cdot,t)\|_{L_q(\Omega)}, \tag{4.14}$$

which furnishes that

$$\sum_{\ell=1}^{M}\Big(\int_0^t (e^{\gamma s}|(\mathbf{u}(\cdot,s),\mathbf{p}_\ell)_\Omega|^p ds\Big)^{1/p} \le C(\epsilon+\mathbf{I}(t)^2). \tag{4.15}$$

Combining (4.11) and (4.15), we have (4.4).

Finally, using (4.4), we show that solutions can be prolonged to any time interval beyond $(0,T)$. Let $r_\pm(\epsilon)=(2M_1)^{-1}\pm\sqrt{(2M_1)^{-2}-\epsilon}$ be the two roots of the quadratic equation: $M_1(x^2+\epsilon)-x=0$. If $0<\epsilon<(2M_1)^{-2}$ then $0<r_-(\epsilon)<r_+(\epsilon)$ and $r_-(\epsilon)=M_1\epsilon+O(\epsilon^2)$ as $\epsilon\to 0+0$. Since $\mathbf{I}(t)\to 0$ as $t\to 0$ and $\mathbf{I}(t)$ is continuous with respect to t as long as solutions exist, there exists an $\epsilon_0\in(0,1)$ such that

$$\mathbf{I}(t)\le r_-(\epsilon)\le 2M_1\epsilon \tag{4.16}$$

for any $t\in(0,T)$ and $\epsilon\in(0,\epsilon_0)$. By (4.5),

$$\|\theta(\cdot,T)\|_{W_q^1(\Omega)}+\|\mathbf{u}(\cdot,T)\|_{B_{q,p}^{2(1-1/p)}(\Omega)}+\|\omega(\cdot,T)\|_{W_q^1(\Omega)}\le M_2\epsilon\le M_2 \tag{4.17}$$

with some constant M_2 independent of ϵ. By (4.7), $\|\theta(\cdot,T)\|_{L_\infty(\Omega)}\le C\|\theta(\cdot,T)\|_{W_q^1(\Omega)}\le CM_2\epsilon$, so that choosing ϵ so small that $CM_2\epsilon<(1/3)\rho_*$, we have

$$\frac{2}{3}\rho_*<\rho_*+\theta(x,T)<\frac{4}{3}\rho_*. \tag{4.18}$$

We consider the nonlinear equations:

$$
\begin{cases}
\partial_t\bar\theta+\rho_*\mathrm{div}\,\bar{\mathbf{u}}=\tilde f(\bar\theta,\bar{\mathbf{u}},\bar\omega) & \text{in } \Omega\times(T,T+T_1)\\
\rho_*\partial_t\bar{\mathbf{u}}-\mathrm{Div}\,\mathbf{S}(\bar{\mathbf{u}})+P'(\rho_*)\nabla\bar\theta-\beta\mathrm{Div}\,\bar\omega=\tilde{\mathbf{g}}(\bar\theta,\bar{\mathbf{u}},\bar\omega) & \text{in } \Omega\times(T,T+T_1)\\
\partial_t\bar\omega+\gamma\bar\omega-\delta\mathbf{D}(\bar{\mathbf{u}})=\tilde{\mathbf{L}}(\bar\theta,\bar{\mathbf{u}},\bar\omega) & \text{in } \Omega\times(T,T+T_1)\\
(S(\bar{\mathbf{u}})-P'(\rho_*)\bar\theta\mathbf{I}+\beta\bar\omega)\mathbf{n}=\tilde{\mathbf{h}}(\bar\theta,\bar{\mathbf{u}},\bar\omega) & \text{on } \Gamma_1\times(T,T+T_1)\\
\bar{\mathbf{u}}=0 & \text{on } S\times(T,T+T_1)\\
(\bar\theta,\bar{\mathbf{u}},\bar\omega)|_{t=T}=(\theta(\cdot,T),\mathbf{u}(\cdot,T),\omega(\cdot,T)) & \text{in } \Omega
\end{cases}
\tag{4.19}
$$

which is the corresponding equations to main problem for time interval $(T,T+T_1)$. Here, $\tilde f(\bar\theta,\bar{\mathbf{u}},\bar\omega)$, $\tilde{\mathbf{g}}(\bar\theta,\bar{\mathbf{u}},\bar\omega)$, $\tilde{\mathbf{L}}(\bar\theta,\bar{\mathbf{u}},\bar\omega)$ and $\tilde{\mathbf{h}}(\bar\theta,\bar{\mathbf{u}},\bar\omega)$ are nonlinear functions defined by replacing θ, $\mathbf{u}$, ω and $\int_0^t\nabla\mathbf{u}ds$ by $\bar\theta$, $\bar{\mathbf{u}}$, $\bar\omega$ and $\int_0^T\nabla\mathbf{u}ds+\int_T^t\nabla\bar{\mathbf{u}}ds$ in (1.12), respectively. Since $\int_0^T\|\nabla\mathbf{u}(\cdot,s)\|_{L_\infty}ds\le C\epsilon$ as follows from (4.7) and (4.16), employing the same argumentation as in the proof of the local well-posedness for problem (1.10) due to Maryani [3] or the local well-posedness for the compressible barotropic viscous fluid flow due to Enomoto, von Below and Shibata [2], we can choose positive numbers ϵ and T_1 so small that problem (4.19) admits unique solutions $\bar\theta$, $\bar{\mathbf{u}}$, and $\bar\omega$ with

$$
\begin{aligned}
&\bar\theta\in W_p^1((T,T+T_1),W_q^1(\Omega)),\quad \bar{\mathbf{u}}\in L_p((T,T+T_1),W_q^2(\Omega)^N)\cap W_p^1((T,T+T_1),L_q(\Omega)^N),\\
&\omega\in W_p^1((T,T+T_1),W_q^1(\Omega))
\end{aligned}
$$

satisfying the estimates

$$\int_T^{T+T_1} \|\nabla \bar{\mathbf{u}}(\cdot,t)\|_{L_\infty(\Omega)} dt \leq \sigma/2, \quad \frac{1}{3}\rho_* < \rho_* + \bar{\theta}(x,t) < \frac{5}{3}\rho_* \quad ((x,t) \in \Omega \times (T, T+T_1)). \tag{4.20}$$

If we define θ_1, ω_1 and $\mathbf{u}_1$ by

$$\theta_1(x,t) = \begin{cases} \theta(x,t) & \text{for } 0 < t < T, \\ \bar{\theta}(x,t) & \text{for } T < t < T+T_1, \end{cases} \quad \mathbf{u}_1(x,t) = \begin{cases} \mathbf{u}(x,t) & \text{for } 0 < t < T, \\ \bar{\mathbf{u}}(x,t) & \text{for } T < t < T+T_1, \end{cases}$$
$$\omega_1(x,t) = \begin{cases} \omega(x,t) & \text{for } 0 < t < T, \\ \bar{\omega}(x,t) & \text{for } T < t < T+T_1, \end{cases}$$

then θ_1, ω_1 and $\mathbf{u}_1$ solve (1.10) in $(0, T+T_1)$ and

$$\theta_1 \in W^1_p((0,T+T_1), W^1_q(\Omega)), \quad \mathbf{u}_1 \in L_p((0,T+T_1), W^2_q(\Omega)^N) \cap W^1_p((0,T+T_1), L_q(\Omega)^N),$$
$$\omega_1 \in W^1_p((0,T+T_1), W^1_q(\Omega)).$$

Moreover, by (4.16), (4.7) and (4.20) we have $\frac{1}{3}\rho_* < \rho_* + \theta_1(x,s) < \frac{5}{3}\rho_*$ and

$$\sup_{0<t<T+T_1} \| \int_0^t \nabla \mathbf{u}_1(\cdot,s)\, ds\|_{L_\infty(\Omega)} \leq \int_0^T \|\nabla \mathbf{u}(\cdot,s)\|_{L_\infty(\Omega)} ds + \int_T^{T+T_1} \|\nabla \bar{\mathbf{u}}(\cdot,s)\|_{L_\infty(\Omega)} ds$$
$$\leq M_3\epsilon + \sigma/2$$

with some constant M_3 independent of ϵ. Choosing $\epsilon > 0$ so small that $M_3\epsilon \leq \sigma/2$, we see that θ_1 and $\mathbf{u}_1$ satisfy (4.3) replacing T by $T+T_1$. Therefore, we can prolong θ, $\mathbf{u}$, and ω to $(0, T+T_1)$. It follows from (4.17) that T_1 is independent of ϵ, so that we can prolong θ, $\mathbf{u}$, and ω to time interval $(0,\infty)$ finally with $\mathbf{I}(\infty) \leq r_1(\epsilon)$, which completes the proof of the existence part of Theorem 1.1. But, the uniqueness follows from the local in time unique existence theorem (Theorem 1.1), which completes the proof of Theorem 1.3.

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